Vector Charmonium Meson-Hybrid Mixing: a Field Theoretical Analysis

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Abstract

Quantum chromodynamics (QCD) is the quantum field theory describing strong interactions. Hybrids in QCD, which are bound states consisting of a charm and an anticharm quark with a constituent gluon, have been theorized for some time. In this thesis we begin to explore the idea that perhaps these particles exist as quantum mechanical superpositions of hybrid and pure mesonic states (which are bound states consisting of a quark and an antiquark). In particular, we will be interested in vector charmonium (charm-anticharm) meson-hybrid mixing. Here we do a field theoretical analysis of charmonium meson-hybrid mixing in the $J^{PC} = 1^{--}$ channel; the two point cross-correlator has been calculated to leading order in the strong coupling (α_s). We include the perturbative, four dimensional and six dimensional condensate contributions. The perturbative contribution was found to contain non-polynomial divergences which were addressed through the introduction of operator mixing. The results of this calculation are presented in a form that is ready for a QCD sum rules analysis.

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List of Abbreviations

QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
LHS	Left Hand Side
RHS	Right Hand Side
SSB	Spontaneous Symmetry Breaking
TOP	Time-Ordered Product
NOP	Normal-Ordered Product
VEV	Vacuum Expectation Value
dim-reg	dimensional regularization
DIS	Deep Inelastic Scattering

Chapter 1

Introduction

We begin with a brief overview of the topics in modern particle physics that will be important in understanding our exploration of charmonium meson-hybrid mixing. Once we have reviewed the key concepts upon which the work in this thesis will be based, we will do a quick survey of the literature that motivated the question of charmonium meson-hybrid mixing.

1.1 The Quark Model

By the 1950s, the idea that atoms were made up of a positively charged nucleus and some number of negatively charged electrons bound together by the electromagnetic force, had been around for decades. The fact that these nuclei were themselves made up of more fundamental nucleons, positively charged protons and neutral neutrons, had also been known for decades.

The discovery of the substructure of the nucleus originally raised questions about how all of these positively charged protons could remain bound, considering the electromagnetic repulsion they should experience. This problem was solved by introducing the strong nuclear force, which acted as the binding force that provided the attraction for these nucleons. It was understood that this force would need to be strong enough to overpower the electromagnetic repulsion of the protons over short ranges, but it would also need to fall off quickly as distances increased beyond the size of the atomic nucleus. A subatomic particle had also

been theorized to mediate this strong nuclear force.

By the early 1950s, pions had been discovered; these pions fit neatly into the particle physics of that era as they could act as the carrier particles of the strong nuclear force. In addition to these pions, other particles were being discovered around this time; such as the kaon (or K meson) and the lambda baryon. As time went on through the 1950s and early 1960s, more and more of these subatomic particles were being discovered. Around this time the term "hadron" was coined to serve as a blanket term for all of these particles. So many hadrons were being discovered that it was quickly becoming obvious that they could not be fundamental. A theory that explained all of these hadrons and their substructure was needed and this is where the quark model came in.

The quark model, which was independently proposed by Murray Gell-Mann [2] and George Zweig [3, 4] in 1964, is a classification scheme for hadrons in terms of their valence quarks. That is to say that the quark model sorts hadrons according to their quantum numbers, which are determined by their constituent quarks and antiquarks.

These quarks come in one of six flavours which are listed here in ascending order by mass: {up, down, strange, charm, bottom, top}. All of these are spin-$\frac{1}{2}$ fermions, and each of them has an intrinsic fractional electric charge (Q). For the up, charm and top quarks, $Q = \frac{2}{3}e$, and for the down, strange and bottom quarks, $Q = -\frac{1}{3}e$, where e is the elementary charge. Combinations of these quarks and their antiquark counterparts in bound states give us the hadrons. In the quark model, hadrons can be broken down into two categories: baryons, which are a bound state of three quarks (or antiquarks), and mesons which are a bound state of a quark and an antiquark see Figure 1.1.

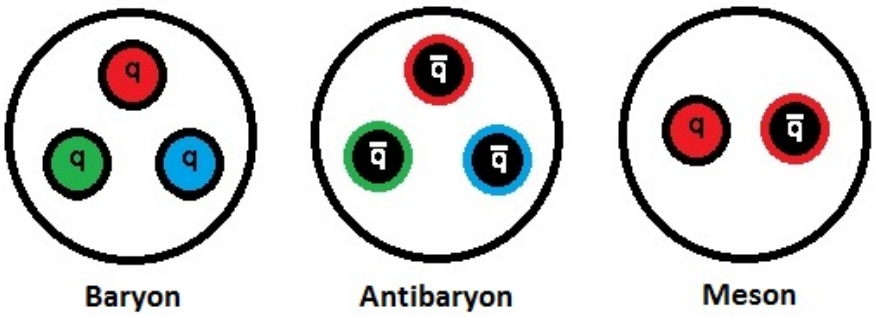

Figure 1.1: Hadrons in the quark model

1.1.1 Quark Model Mesons

In the quark model, mesons are described as a bound state of a quark-antiquark pair. As we know that all quarks are spin-$\frac{1}{2}$ fermions, we know that mesons must have intrinsic spins $\vec{S}$ where our quantum numbers are $s = 0$ or $s = 1$. We can then write the total angular momentum $\vec{J}$ of the bound state as follows:

$$\vec{J} = \vec{S} + \vec{L} \tag{1.1}$$

where $\vec{L}$ is the orbital angular momentum. Here $\vec{L} = 0$ corresponds to the ground state of our meson with higher values of $\vec{L}$ (increasing in integer steps) correspond to excited states. Quantization of angular momentum gives $L^2 = l(l+1),\ l \in \{0, 1, 2, ...\}$. Note that for the entirety of this thesis we will be using natural units where $\hbar = c = 1$.

There are two other quantum numbers that will be useful to us in classifying mesons. These are parity (P) and c-parity (C), which for mesons can be written as functions of angular momentum and spin quantum numbers as follows:

$$P = (-1)^{l+1}, \qquad C = (-1)^{l+s}. \tag{1.2}$$

Collectively we refer to the values J, P and C as a particle's J^{PC} which gives us a natural

way of classifying many particles. It can be shown that, given the above restrictions, we can construct J^{PC}s for mesons where $J^{PC} \in \{0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 1^{++}, 2^{--}, \dots\}$ where any other J^{PC}s not appearing in this list would be referred to as exotic quantum numbers for mesons. The work done in this thesis will focus exclusively on particles with $J^{PC} = 1^{--}$. Particularly, we will be interested in charmonium states which are bound states consisting of a charm and an anticharm quark such as the J/ψ and other excited charmonium states with appropriate quantum numbers.

1.1.2 Successes of the Quark Model

The quark model was very successful at classifying the hadrons known in that era as well as sorting them into the various geometric patterns of the eightfold way. In addition to giving us a classification scheme for the hadrons and helping explain their substructure, the quark model was also able to predict the existence of new hadrons such as the Ω^-. The quark model enjoyed many other successes including explaining mass splitting between mesons within their respective diagrams, explaining and predicting the magnetic moments of mesons and baryons and explaining why there are no spin-1 baryons among several others. The successes of the quark model are well documented in the literature and easy to find for any reader that may be interested in exploring further. However, for all of its successes, the quark model had a couple of problems.

One major problem with the early quark model became clear when it was realized that the Δ^{++} baryon, with its intrinsic spin $S = \frac{3}{2}$ and orbital angular momentum $L = 0$, consisted of three up quarks with parallel spin. It was realized that this would lead to a totally symmetric wave function, violating the Pauli exclusion principle. The best solution to the problem turned out to be the ad hoc introduction of a new quantum number carried by quarks that would come to be known as color. Each quark would now carry a color red, green or blue and antiquarks would carry anti-red, anti-green or anti-blue. The color portion of this wave function would then be totally antisymmetric. The addition of this new quantum number gave us the anti-symmetric piece of the wave function which we need to satisfy the

Pauli exclusion principle and thus fix the problem of the Δ^{++}.

Another serious problem was the question of free quarks. If all hadrons are made up of quarks, why is it that free quarks are never observed? Another ad hoc addition to the quark model was needed to address this issue. It was suggested that if free quarks are not observed they must be confined to these hadrons. Even if these hadrons are collided at sufficient energies to liberate a quark the free quark would never be observed; instead these quarks would immediately undergo hadronization (which I discuss briefly below) and our detectors would only pick up jets of hadrons emerging from the collision event. This phenomenon became known as quark confinement. It was also realized that perhaps this confinement could be characterized in terms of color since all bound states had been observed to be color singlets, and so we refer to confinement as color confinement.

Perhaps the way in which I have chosen to write about the quark model makes these topics seem more discretized than they should be. In reality, the quark model evolved slowly throughout the 1960s with individual ideas being developed and incorporated, until it eventually became what we now think of as the quark model. It should also be mentioned that many of the ideas in the quark model were incorporated into and influenced the development of quantum chromodynamics which is what we will be discussing next.

1.2 Quantum Chromodynamics

Quantum chromodynamics (QCD) is the quantum field theory (QFT) of strong interactions. In essence a QFT can be described as a mathematical framework which allows for the unification of special relativity and quantum mechanics by allowing for the creation and annihilation of particles.

By the late 1960s and early 1970s quantum electrodynamics (QED) was already well established as the QFT of electrodynamics. QED mathematically describes all phenomena involving electrically charged particles interacting by means of photon exchange. The successes of QED, coupled with the insights gained from the quark model, led to attempts to

form a QFT of strong interactions; this theory would become known as QCD.

It was important for QCD to be able to be able to explain experimental results such as deep inelastic scattering (DIS) [5], as well as reproduce the known results from the quark model. It should be mentioned that DIS is an experiment in which protons (or any hadron for that matter) are bombarded with very high energy electrons in an attempt to resolve the substructure of these protons. It was experiments like this that provided the first convincing evidence that quarks were, in fact, real particles and not simply a mathematical construct of the theory as some had previously believed them to be. To match some of the results of these DIS experiments, it was understood that QCD would need to be an asymptotically free theory. An asymptotically free theory can be described as a theory for which the strength of interactions between particles becomes asymptotically weaker as energy increases and distance decreases.

It was shown that in 4-dimensions, the only asymptotically free renormalizable gauge theories were the class of theories known as Yang-Mills theories [6]. These theories are invariant under local SU(N) transformations. It was also realized that the theory would need to exhibit $SU(3)_{color}$ symmetry, as quarks of different colors are indistinguishable from one another. This indicated that QCD should be constructed as an SU(3) Yang-Mills theory. Furthermore Yang-Mills theories are gauge theories, and as such our theory must introduce mediating vector bosons which now must carry our color charge as we have ($N \geq 2$). These vector bosons became known as gluons. There is a rich history surrounding the development of QCD however we will focus on some of the key concepts of the theory as they are understood today before we move on.

1.2.1 Key Concepts in QCD

Where the strong force was once thought of as the force holding the nucleus together, it is now understood that the strong force is actually the force acting on the color charged particles holding hadrons together. It is just the residual effect of this force that holds the nucleus together, which can almost be thought of as a dipole force in an electromagnetic

context.

As we discussed earlier, all quarks are thought of as having one of three colors {Red, Green, Blue} and all antiquarks are {Anti-Red, Anti-Green, Anti-Blue}. QCD has now introduced gluons which have some combination of color and anti-color, and hadrons are understood to exist only as color singlet bound states of these particles in this context.

At close range, quarks are bound loosely inside a hadron. As more energy is put into the system and distances increase, we reach a point where it is more energetically economical to produce new hadrons than to continue to have the distance between these bound quarks grow. As such, hadron jets are produced and free quarks are never observed. This process is what we call hadronization, which was mentioned above.

It is important to note that one strong contrast between QED and QCD is that, unlike photons, gluons (the force carrying vector bosons of QCD) carry color charge and therefore should interact strongly. This will be an important characteristic of gluons for our work in this thesis.

1.2.2 Hybrids in QCD

If we now revisit the discussion we had in Section 1.1 regarding the composition of mesons and their J^{PC}s, we can see that QCD allows for a slightly looser restriction on the makeup of hadrons, and it also supplies us with more building blocks (gluons). In the quark model, we were only able to build hadrons out of either: three quarks, three antiquarks or a quark/antiquark pair. QCD requires only that our bound states be colorless, and it supplies us with gluons as another potential constituent of our hadrons. Turning our attention back to bound states containing a quark and an antiquark, and considering only the colorless bound state restriction, there is no reason one could not build a bound state out of a quark and an antiquark and a gluon. Such particles have been theorized for some time and are known as hybrids.

Discovering a meson like bound state with what would be exotic quantum numbers for a

meson, would point strongly to the existence of hybrids as it would be difficult to explain these quantum numbers without the additional gluonic degree of freedom. Experimental efforts to find these hybrids with exotic quantum numbers continue with some promising candidates emerging, however results are still inconclusive [7].

For the purpose of our explorations, we will be interested in charmonium hybrids with $J^{PC} = 1^{--}$. As we are dealing with non-exotic quantum numbers with regard to conventional mesonic J^{PC}s, this makes finding such particles even more difficult. Some findings that might suggest that these hybrids exist would be the overpopulation of resonances for a particular J^{PC} in a particular spectrum. Also, branching ratios that differ significantly from those predicted by conventional mesonic models would also suggest that hybrids may be present. Detection of these non-exotic hybrids could be further complicated by the fact that they may mix with conventional mesonic states. It is this last point about potential mixing that we will explore farther in this thesis.

1.2.3 The QCD Lagrangian

For now, let us turn our attention to the fundamental quantity of QCD in its Lagrangian formulation; the Yang-Mills Lagrangian which can be written as follows:

$$\mathcal{L}_{\mathrm{QCD}}(x) = -\frac{1}{4}\left(G^a_{\mu\nu}(x)\right)^2 + \sum_F \bar{Q}_F(x)\left(i\not{D} - m_F\right)Q_F(x) \tag{1.3}$$

where

$$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g_s f^{abc} A^b_\mu(x) A^c_\nu(x) \tag{1.4}$$

is a gluon field strength tensor and

$$\not{D} = D^\mu \gamma_\mu = (\partial^\mu - i g_s t^a A^\mu_a(x))\gamma_\mu \tag{1.5}$$

is a slashed covariant derivative.

We will take a moment to go over some of the notation here as much of it will be used repeatedly throughout this thesis. Here the Qs and the As are quark and gluon fields respectively, the Fs are quark flavour indices, but as all of our work will deal with just one flavour of quark (the charm quark) the sum and these indices can be dropped. The m indicates the mass of the particle, g_s is our coupling constant, t^a is a generator of $SU(3)$ and f^{abc} is a totally antisymmetric structure constants. Finally, any slashed variable is understood to employ Feynman slash notation, such as $\not{D} = D^\mu \gamma_\mu$.

We can now expand (1.3) and separate out the free terms from the interaction terms in our Lagrangian. In doing so and by suppressing the arguments, this gives us

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_0 + \frac{g_s}{2}\overline{Q}\gamma^\sigma \lambda^a Q A^a_\sigma - g_s f^{abc}(\partial_\rho A^a_\sigma)A^{\rho b}A^{\sigma c} - \frac{g_s^2}{4}(f^{eab}A^a_\rho A^b_\sigma)(f^{ecd}A^{\rho c}A^{\sigma d}) \qquad (1.6)$$

where $\mathcal{L}_0$ is our free Lagrangian and the $\lambda^a = 2t^a$ are Gell-Mann matrices. Having done this expansion, we could now construct the Feynman rules for the theory. By inspection, we can see that the first term will lead to a quark gluon vertex where the second and third terms will give us three and four gluon vertices respectively. Details about these vertices, as well as more discussion on this Lagrangian, can be found in many QFT texts. For our purposes we will be interested only in the terms in the interaction Lagrangian that will contribute to our calculation at two loops. I will postpone isolating this term and writing the expressions for the propagators until Chapter 2, where they will fit neatly into the flow of our calculation. For now, let us just note that these vertices highlight one of the main differences between QED and QCD; because here we have our vector bosons able to have self-interactions. Moreover, all of these vertices share the same coupling strength. These facts hint that thinking about hybrids with their explicit gluonic degree of freedom is perhaps a sensible thing to do.

An important side note; thus far we have been somewhat careful about the position of indices, this will not be the case in general throughout the thesis. Generally, indices will be raised or

lowered purely out of convenience, this will not affect our calculation at all and will serve to make the notation slightly more aesthetically pleasing.

1.2.4 The Correlation Function

Most of the work done in this thesis will be focused on calculating a specific quantity known as a correlation function. Note that throughout this thesis I will use the terms correlator, cross-correlator and correlation function interchangeably to refer to the quantity we are calculating. This quantity can perhaps most simply be described as the vacuum expectation value of the time ordered product of fields. At its simplest, a two point correlation function such as

$$\langle 0| \, T[\phi(x)\phi(y)] \, |0\rangle \tag{1.7}$$

in a free field theory, can be interpreted as the propagation amplitude for the particle from y to x. Here $\phi(x)$ and $\phi(y)$ are simple field operators; T is the time-ordering operator and $\langle 0| \, |0\rangle$ represent the vacuum in our free theory. In our case, these field operators will be replaced with composite operators containing several fields. These composite operators, which we call currents, will be selected specifically to probe the hadronic states we are interested in examining; namely charmonium hybrid and pure charmonium. These currents must be color singlets and they must have the appropriate quantum numbers for the states we want to probe; $J^{PC} = 1^{--}$ in this case. We will get into the details of these currents when we start the calculation, but in principle (1.7) would now become

$$\langle \Omega| \, T[\, j^{\mu}_{(m)}(x) \; j^{\nu}_{(h)}(0)] \, |\Omega\rangle \tag{1.8}$$

where $j^{\mu}_{(m)}(x)$ and $j^{\nu}_{(h)}(0)$ are our pure meson current at x and our hybrid current at 0 respectively (in the Heisenberg picture when written this way) and $\langle \Omega| \, |\Omega\rangle$ is our QCD vacuum. This time-ordered product can be written in the interaction picture as follows:

$$\Pi^{\mu\nu}(q) = i\int d^d x\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\; j^{\nu}_{(h)}(0)\; e^{i\int dy \mathcal{L}_{int}(y)}]|\Omega\rangle \tag{1.9}$$

where in actuality the equality here is approximate and will require us to manually discard any disconnected diagrams. Here, $e^{i\int dy \mathcal{L}_{int}(y)}$ contains QCD interaction Lagrangian terms we discussed in (1.6), and we have taken the momentum space transform of the correlation function. Notice that the integral is d dimensional as we will be using dimensional regularization which we will discuss below. Note also, that all integrals in this thesis where bounds are not explicitly written are taken to be over the full range of their dummy variables.

In this section, we have briefly gone over how we could write our correlator but we also need to understand what it is and why we want to calculate it. Without introducing several new terms and ideas, perhaps the easiest way to describe the correlator is to describe how it will be used. The correlator is the quantity that contains all this information about our system, which we will feed into the dispersion relation.

1.2.5 The Dispersion Relation

Once our correlator has been sufficiently simplified and broken up into its scalar and vector portions (as we describe in Section 2.1.5), we will feed these into the dispersion relation. The dispersion relation can be written as

$$\Pi(Q^2) = \frac{Q^6}{\pi}\int_{M_Q^2}^{\infty}\frac{\mathrm{Im}[\Pi(t)]}{t^3(t+Q^2)}dt + \cdots, \quad Q^2 > 0, \quad Q^2 = -q^2 \tag{1.10}$$

where the $\cdots$ represent subtraction constants; collectively a second degree polynomial in q^2, and the $\mathrm{Im}[\Pi(t)]$ is our hadronic spectral function. This dispersion relation is an expression of quark/hadron duality. On the left hand side, we have our correlator calculated in terms of quarks and the dispersion relation relates this quantity to the hadronic spectral function on the right, which contains information about hadrons. In practice, this allows us to calculate in terms of quarks and make predictions about hadrons. Our hadronic spectral function could,

in principle, be written in terms Dirac delta functions and Heaviside step functions which would represent the resonances we would expect to see in the hadronic spectrum. In turn, this would allow us to extract the physical mass of our theorised particle. This, however, is farther than we will be taking the calculation in this thesis. For now, we will be content to solve for the correlator in a form which is ready for the dispersion relation and we will save the dispersion relation and sum rules analysis [8] for a later project.

1.2.6 The Operator Product Expansion

We now turn our attention back to our correlator and look at some of the tools we will need to simplify it. Substituting the appropriate currents into (1.9) gives us the equation we will now be working with. Remembering that each of the three terms on the right hand side of (1.9) will contain some number of operators in the form of quark and gluon fields, we will need some tools to help us evaluate the products of these non-local fields and their vacuum expectation values (VEV)s. The operator product expansion (OPE) states that, for a product of operators $\mathcal{O}_1$ and $\mathcal{O}_2$ acting at spacetime coordinates x and y, we can write:

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \to \sum_n C_{12}^n(x)\mathcal{O}_n(y) \quad \text{as} \quad x \to y \tag{1.11}$$

where the Cs are classical numbered functions known as Wilson coefficients. This, in effect, allows us to write the product of fields at two distinct space-time points in terms of fields at one of those points [9]. In practice, for our calculation, $\mathcal{O}_1$ and $\mathcal{O}_2$ will be our currents and both sides of the equation will be wrapped in a time-ordered product (TOP) and be inside a VEV. In essence, this will relate our correlation function to the expansion that will allow us to simplify it. The right hand side of this expression will give us a series of local VEVs, including a perturbative contribution and non-perturbative terms that will be characterized by condensates, which we will look at briefly after a quick discussion on Wick's theorem.

1.2.7 Wick's Theorem

Wick's Theorem allows us to write the TOP of some collection of operators in terms of the sum of normal-ordered products (NOP)s of those operators, and the NOPs of their contractions. These contracted fields are then written in terms of propagators; we will discuss the valid contraction schemes and appropriate propagators for our specific problem as we get into the calculation in subsequent chapters. Much more detail about this process is available in most QFT texts including [10] but, in essence, we are now able to write:

$$\begin{aligned} T\{\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)\} = \\ = N\{\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n) + \textit{all possible contractions}\} \end{aligned} \tag{1.12}$$

where N indicates a NOP. Each of the terms in the right hand side of (1.12) can be thought of as representing a Feynman diagram. Those terms, where the fields are fully contracted, can be thought of as connected diagrams that could be evaluated using the QCD Feynman rules. These diagrams represent the perturbative contribution to our cross-correlator. The terms where fields remain uncontracted and thus leave us with the VEVs of uncontracted fields are generally taken to be zero. In the case of QCD however, these VEVs can be non-zero due to the complexity of the QCD vacuum and, as mentioned above, these terms will be characterized in terms of condensates. Once evaluated these non-zero VEVs will represent the non-perturbative contribution to our cross-correlator.

1.2.8 Condensates and the QCD Vacuum

As mentioned above, through the application of the OPE and Wick's theorem, we will generate a number of terms that will contain the VEV of uncontracted local operators. In theories with a simpler vacuum, these terms would be taken to be zero. However the QCD vacuum is slightly more complex and results in non-zero values for these VEVs. The complexity of the

QCD vacuum and the non-zero nature of these VEVs are driven by spontaneous symmetry breaking (SSB) in QCD. SSB manifests in a theory where a symmetry associated with the Lagrangian is not shared by the ground state of the theory. Here, our ground state is the QCD vacuum and our Lagrangian is the Yang-Mills Lagrangian. SSB is a topic that is well covered in most QFT texts including [10]. For our purposes it will be enough to understand that these local VEVs are non-zero, and that their numeric values are external inputs to the theory that need to be measured experimentally.

1.2.9 Regularization

Once we have applied the OPE and Wick's theorem to our cross-correlator and gone through the considerable algebra needed to simplify our expression (which we will go over in detail as we review our actual calculation in the coming chapters), we will be left with an expression phrased in terms of an internal momentum space integral. These integrals can be complicated and are generally divergent in four dimensions. The technique we use to deal with these divergent integrals is what is known as dimensional regularization (dim-reg). Dim-reg is perhaps the simplest regularization scheme which preserves the symmetries of QCD; it is widely used and well described in many QFT texts including [10]. Put simply, in dim-reg we promote our divergent four dimensional integral to a d dimensional integral that will converge, and analytically continue the result to other dimensions. In essence, we compute our integral as an analytical function of the dimensionality of spacetime. Our final expression will be phrased in terms of our spacetime dimension d and will have a well defined limit as $d \to 4 + 2\epsilon$; a convention consistent with [11].

In principle, dim-reg gives us a framework within which all of our integrals are convergent but in practice the solving of these integrals can still be quite complicated. Generally, one-loop integrals can be solved without much trouble but two-loop integrals often require the application of recurrence relations to simplify the integrals before they are in a solvable form. We will be using the TARCER [12] Mathematica software package to help in evaluating these two-loop integrals. We will go into more detail about TARCER in Section 2.1.6 but for now

it can be thought of as a tool which will allow us to write complicated two-loop integrals in terms of master integrals with known solutions.

1.2.10 Renormalization

It is important to note that one of the currents we will be using, our hybrid current $j_\nu^{(h)}$ (which is itself a composite operator that we write out in equation (2.4)), is not renormalized. Using an unrenormalized current as we do in this calculation will result in non-polynomial divergences in our final answer. When doing the sum rules analysis of this correlator, polynomial divergences will not pose a problem as they will be eliminated by a Borel transform [8] defined by

$$\hat{\mathcal{B}} = \lim_{\substack{N,Q^2\to\infty \\ \tau=N/Q^2}} \frac{(-Q^2)^N}{\Gamma(N)} \left(\frac{d}{dQ^2}\right)^N. \tag{1.13}$$

However, these non-polynomial divergences will need to be addressed before the correlator is ready for sum rules analysis. To deal with these non-polynomial divergences, we introduce some notation. If we let a square bracket indicate a renormalized quantity and a quantity with no brackets indicate a bare quantity we know that:

$$\left[j_{(h)}^\nu\right] = Z_1 j_{(h)}^\nu + Z_2\mathcal{O}_2 + \cdots + Z_n\mathcal{O}_n. \tag{1.14}$$

In this case, the Z_n are renormalization constants that will depend on what order in g_s we calculate to and the $\mathcal{O}_n$ are composite operators with the same quantum numbers as our $j_{(h)}^\nu$ with dimension less than or equal to that of $j_{(h)}^\nu$ [13]. The ideas about renormalization discussed here are further discussed in [13]. We will make use of these ideas when we come to deal with the non-polynomial divergences in our calculation in Chapter 3.

1.3 Motivations

We have spent the last few sections introducing some of the key theoretical ideas that frame the work done in this thesis. Let us now turn our attention to the experimental and theoretical findings that motivated the question we will be setting up to answer.

We start by looking at the charmonium spectrum and what are known as the XYZ resonances. These XYZ mesons are hadrons that have been detected experimentally and have been seen to decay to final states which consist of a heavy quark-antiquark pair but do not fit neatly into the quark model's $q\bar{q}$ scheme [1]. Specifically, we will be interested in charmonium-like XYZ mesons with $J^{PC} = 1^{--}$ that do not fit into the $c\bar{c}$ mass spectrum i.e. the J/ψ and its radial excitations. The XYZ resonances that fit this description are the $Y(4260)$, $Y(4360)$ and $Y(4660)$. The $Y(4260)$ and $Y(4360)$ were discovered by BaBar in the $\pi^+\pi^- J/\psi$ and $\pi^+\pi^-\psi'$ systems respectively via $e^+e^- \to \gamma_{isr}\pi^+\pi^- J/\psi$ and $e^+e^- \to \gamma_{isr}\pi^+\pi^-\psi'$ [14, 15]. These results were later confirmed by the Belle Collaboration [16, 17] which also found another peak in the $\pi^+\pi^-\psi'$ system, the $Y(4660)$. We can see how these states fit into the charmonium and charmonium like meson spectrum in Figure1.2 (image credit Olsen, Front. Phys. 10 (2015) 101401 [1]).

We can see that these particles fall outside of the quark model description of mesons as all of the 1^{--} $c\bar{c}$ states near their masses have already been assigned to other resonances. It has been suggested that the $Y(4260)$ is a charmonium hybrid state [18]. Also there have been theoretical explorations of $c\bar{c}g$ hybrid currents with $J^{PC} = 1^{--}$ done using QCD sum rules resulting in mass predictions in the range of 3.36 ± 0.15 GeV [19]. These unassigned resonances and the fact that QCD allows for mesonic states with an explicit gluonic degree of freedom have served as the motivation for much of the research into hybrids.

Expanding on some of these ideas, in this thesis we plan to explore the idea that perhaps some of these resonances do not exist as pure mesonic states, but rather as quantum mechanical superpositions of pure and hybrid states. As mentioned previously, we will be interested in $c\bar{c}$ mixing with $c\bar{c}g$ in the $J^{PC} = 1^{--}$ channel. Keeping in mind the findings in [19] where

the $c\bar{c}g$ current with $J^{PC} = 1^{--}$ was predicted to have a mass of 3.36 ± 0.15 GeV, it seems plausible that the reason we do not see this resonance in the charmonium spectrum might be because it exists as a quantum mechanical superposition with the ψ' or perhaps with the ψ''. Said another way, perhaps the resonance we have assigned to the ψ' or the ψ'' is in fact a mix of pure $c\bar{c}$ with $c\bar{c}g$. The first step in exploring this possibility will be calculating the cross-correlator of two currents that will probe these states. To do so we will make use of many of the tools we have discussed in this introduction. A future project will be to take this correlator and use it in a QCD sum rules analysis in order to get at the physical mass and see where in the charmonium spectrum it falls.

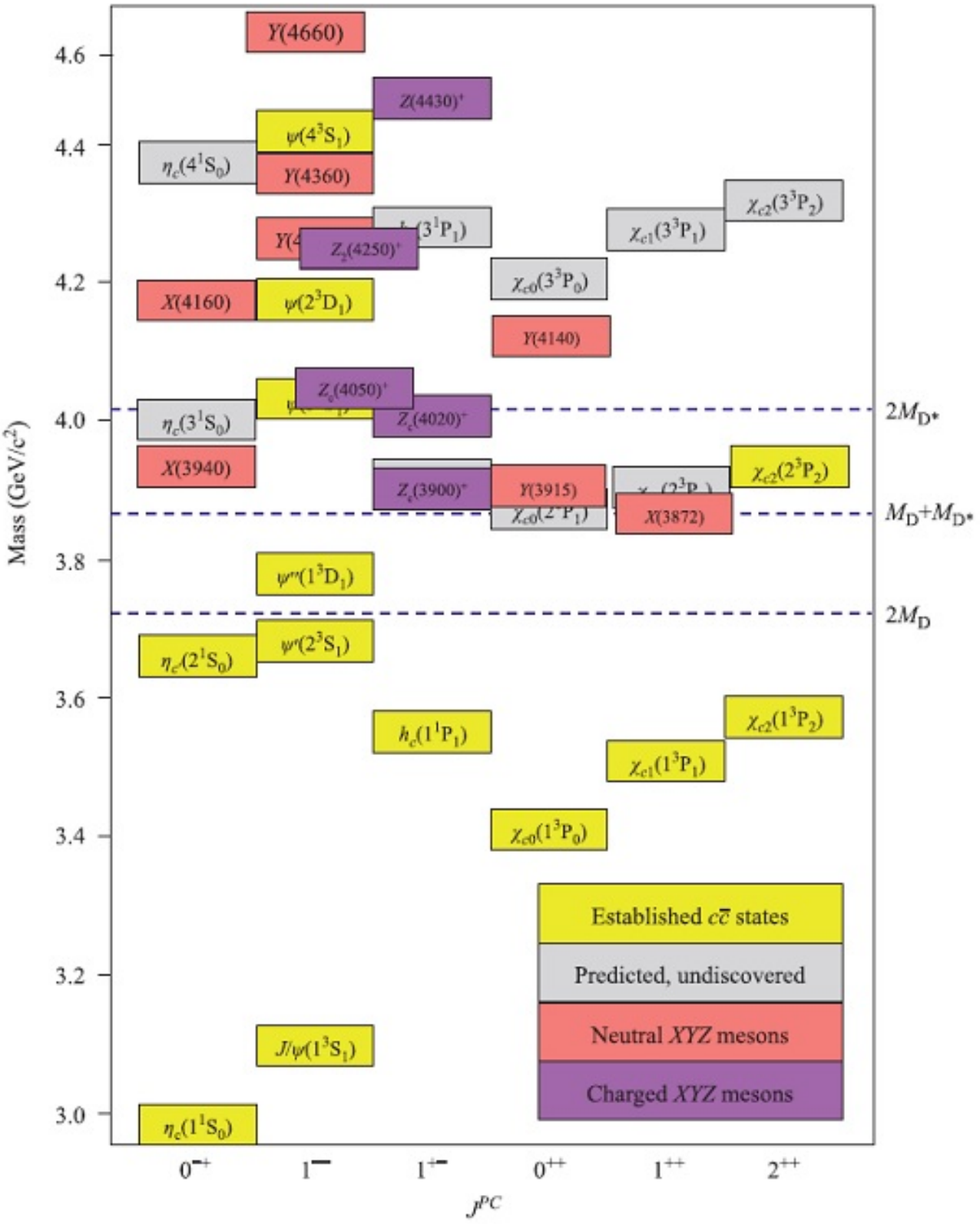

Figure 1.2: The spectrum of charmonium and charmoniumlike mesons, image credit Olsen, Front. Phys. 10 (2015) 101401 [1].

Chapter 2

Correlation Function Calculations

We now begin exploring the specific correlation function we need to calculate. Starting from equation (1.9) we have:

$$\Pi^{\mu\nu}(q) = i\int d^dx\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\; j^{\nu}_{(h)}(0)\; e^{i\int dy\mathcal{L}_{int}(y)}]|\Omega\rangle \tag{2.1}$$

where again, $j^{\mu}_{(m)}(x)$ and $j^{\nu}_{(h)}(0)$ are the pure meson and hybrid currents which we will discuss shortly. The exponential factor $e^{i\int dy\mathcal{L}_{int}(y)}$ is defined as follows:

$$\begin{aligned} e^{i\int dy\mathcal{L}_{int}(y)} &= 1 + i\int d^dz\,\frac{g_s}{2}\,\overline{Q}(z)\gamma^{\sigma}\lambda^b Q(z)A^b_{\sigma}(z) + \ldots \\ &= 1 + i\int d^dz\,\frac{g_s}{2}\,\overline{Q}^{\epsilon}_{n}(z)\gamma^{\sigma}_{nr}\lambda^b_{\epsilon\xi}Q^{\xi}_{r}(z)A^b_{\sigma}(z) + \ldots \end{aligned} \tag{2.2}$$

where $\mathcal{L}_{int}(y)$ represents only the terms in the QCD interaction Lagrangian which will contribute to our calculation at leading order. Note, that the second expression in (2.2) is identical to the first, except that we have now added additional indices to facilitate easier calculation. Note also, that as a general rule, we will take Greek indices on quark and antiquark fields or lambda matrices to represent quark color indices where Greek indices on gamma matrices and gluon fields or gluon field strength tensors will be Lorentz indices. Latin indices on lambda matrices and gluon fields or gluon field strength tensors will represent gluon color, and Latin indices on quark and antiquark fields or gamma matrices will be

Dirac indices. Expanding this exponential results in the infinite series we see in (2.2). Every subsequent term in this expansion corresponds to higher and higher order contributions to our correlator. For the purpose of this calculation we truncate this series after the second term. Subsequent terms in the series will go like higher powers of $\frac{\alpha_s(M_\tau)}{\pi} \sim 0.104$ [20] so we are justified in ignoring higher order terms in this calculation.

Later, we will show that the first term in the expansion (2.2) for the currents we will be using contributes nothing to our correlator. The reason being that the diagram corresponding to this term contains a massless tadpole and thus must be zero. This will leave us only the second term in the series to deal with. We will expend most of our effort calculating the perturbative and non-perturbative contributions to the correlator from this second term.

For this calculation we will be using the two currents that we briefly introduced in Chapter 1. We now write these currents out explicitly. Together they will allow us to probe the mixing we are interested in. The first of these currents takes the following form:

$$\begin{aligned} j^{\mu}_{(m)}(x) &= \overline{Q}(x)\gamma^{\mu}Q(x) \\ &= \overline{Q}^{\alpha}_{i}(x)\gamma^{\mu}_{ij}Q^{\alpha}_{j}(x). \end{aligned} \tag{2.3}$$

This is the current that will probe mesonic states with $J^{PC} = 1^{--}$ and can be found in [21]. We will refer to (2.3) as our mesonic current from this point forward. Our second current takes the form:

$$\begin{aligned} j^{\nu}_{(h)}(0) &= \frac{g_s}{2}\overline{Q}(0)\gamma^{\rho}\gamma^{5}\lambda^{a}Q(0)\frac{1}{2}\epsilon^{\nu\rho\omega\zeta}G^{a}_{\omega\zeta}(0) \\ &= \frac{g_s}{2}\overline{Q}^{\gamma}_{k}(0)\gamma^{\rho}_{ks}\gamma^{5}_{sm}\lambda^{a}_{\gamma\delta}Q^{\delta}_{m}(0)\frac{1}{2}\epsilon^{\nu\rho\omega\zeta}G^{a}_{\omega\zeta}(0). \end{aligned} \tag{2.4}$$

This current will probe hybrid states with $J^{PC} = 1^{--}$ and will be referred to as the hybrid current from now on. This current and other hybrid currents can be found in [22]. We need

to keep in mind that this hybrid current is not renormalized and this will lead to issues that we will deal with in Chapter 3.

With the currents (2.3) and (2.4) inserted into equation (2.1), we will be probing the mixed state we have set out to investigate. Substituting (2.4) and (2.3) into (2.1) and keeping only the second term in the (2.2) expansion, the correlator we wish to calculate, after a bit of simplification, takes the following form:

$$\Pi^{\mu\nu}(q) = -\frac{g_s^2}{8}\int d^dx \int d^dz\; e^{iq\cdot x}\,\gamma^\mu_{ij}\,\gamma^\rho_{ks}\,\gamma^5_{sm}\,\gamma^\sigma_{nr}\,\lambda^a_{\gamma\delta}\,\lambda^b_{\epsilon\xi}\,\epsilon^{\nu\rho\omega\zeta} \\ \langle\Omega|T[\,\overline{Q}^\alpha_i(x)\,Q^\alpha_j(x)\,\overline{Q}^\gamma_k(0)\,Q^\delta_m(0)\,\overline{Q}^\epsilon_n(z)\,Q^\xi_r(z)\,G^a_{\omega\zeta}(0)\,A^b_\sigma(z)\,]|\Omega\rangle. \tag{2.5}$$

Now that we have an expression for the correlator we want to calculate, let us take a look at the perturbative contribution.

2.1 Perturbative Contribution

Let us start our perturbative calculation by looking at the time ordered product in (2.5), namely:

$$T[\,\overline{Q}^\alpha_i(x)\,Q^\alpha_j(x)\,\overline{Q}^\gamma_k(0)\,Q^\delta_m(0)\,\overline{Q}^\epsilon_n(z)\,Q^\xi_r(z)\,G^a_{\omega\zeta}(0)\,A^b_\sigma(z)\,]. \tag{2.6}$$

We will start by splitting this expression into two time ordered products; one containing our fermionic fields and the other our bosonic fields. This simplification is facilitated by the fact that bosonic and fermionic fields commute. After we have done this, we will want to expand out our gluon field strength tensor G into its constituent gluon fields using (1.4), which has been reproduced here with appropriate indices for the reader's convenience:

$$G^a_{\omega\zeta}(0) = \partial_\omega A^a_\zeta(0) - \partial_\zeta A^a_\omega(0) + g_s f^{abc} A^b_\omega(0) A^c_\zeta(0). \tag{2.7}$$

This splitting of the time ordered product and expansion of G will give us

$$\begin{aligned} T[\,\overline{Q}^\alpha_i(x)\, Q^\alpha_j(x)\, \overline{Q}^\gamma_k(0)\, Q^\delta_m(0)\, \overline{Q}^\epsilon_n(z)\, Q^\xi_r(z)\,]\times \\ \times T[\,\Big(\partial_\omega A^a_\zeta(0) - \partial_\zeta A^a_\omega(0) + g_s f^{abc} A^b_\omega(0) A^c_\zeta(0)\Big)\, A^b_\sigma(z)\,]. \end{aligned} \tag{2.8}$$

We can now continue by applying Wick's theorem to these TOPs. As we discussed in Chapter 1, this will give us the normal-ordered sum of all possible combinations of contractions. We will do this in two stages, first for all fermionic fields then for all bosonic fields.

2.1.1 Wick's Theorem

As we are looking only for the perturbative contribution at this point, any terms that are not fully contracted will be ignored. As such, the only non-zero contraction schemes emerging from an application of Wick's theorem on the fermionic fields are

$$N[\,\overline{Q}^\alpha_i(x)\, Q^\alpha_j(x)\, \overline{Q}^\gamma_k(0)\, Q^\delta_m(0)\, \overline{Q}^\epsilon_n(z)\, Q^\xi_r(z)\,] \tag{2.9}$$

and

$$N[\,\overline{Q}^\alpha_i(x)\, Q^\alpha_j(x)\, \overline{Q}^\gamma_k(0)\, Q^\delta_m(0)\, \overline{Q}^\epsilon_n(z)\, Q^\xi_r(z)\,]. \tag{2.10}$$

We can now turn our attention to the bosonic portion of (2.8). Remembering again that only fully contracted terms are included in our perturbative contribution, we see immediately that the third term in the product must be ignored. If we then expand the remaining product,

we are left with two normal ordered products each of which can only be contracted in one way as follows:

$$N[\,\partial_\omega \overbrace{A^a_\zeta(0)\,A^b_\sigma}(z)\,] - N[\,\partial_\zeta \overbrace{A^a_\omega(0)\,A^b_\sigma}(z)\,]. \tag{2.11}$$

Now is an opportune time to go back to the point we made in the preamble to this chapter while discussing what terms we were going to keep in (2.2). We mentioned that we could drop the first term (the 1) as it would go to zero, and now we can see why. If there was no $A(z)$ factor to contract with, all of the bosonic contractions that we could build out of the bosonic portion of (2.8) would go to zero.

Returning to the main calculation; we now multiply our results from (2.9) and (2.10) into (2.11), and we can then see the two contributing contraction schemes which emerge from (2.6):

$$N[\,\overbrace{\overline{Q}^\alpha_i(x)\,Q^\alpha_j(x)\,\overline{Q}^\gamma_k(0)\,Q^\delta_m(0)\,\overline{Q}^\epsilon_n(z)\,Q^\xi_r}(z)\,]\Big(N[\,\partial_\omega \overbrace{A^a_\zeta(0)\,A^b_\sigma}(z)\,] - N[\,\partial_\zeta \overbrace{A^a_\omega(0)\,A^b_\sigma}(z)\,]\Big) \tag{2.12}$$

and

$$N[\,\overbrace{\overline{Q}^\alpha_i(x)\,Q^\alpha_j(x)\,\overline{Q}^\gamma_k(0)\,Q^\delta_m(0)\,\overline{Q}^\epsilon_n(z)\,Q^\xi_r}(z)\,]\Big(N[\,\partial_\omega \overbrace{A^a_\zeta(0)\,A^b_\sigma}(z)\,] - N[\,\partial_\zeta \overbrace{A^a_\omega(0)\,A^b_\sigma}(z)\,]\Big) \tag{2.13}$$

which if substituted back into (2.5) would correspond to the two Feynman diagrams represented in Figure 2.1 and Figure 2.2 respectively.

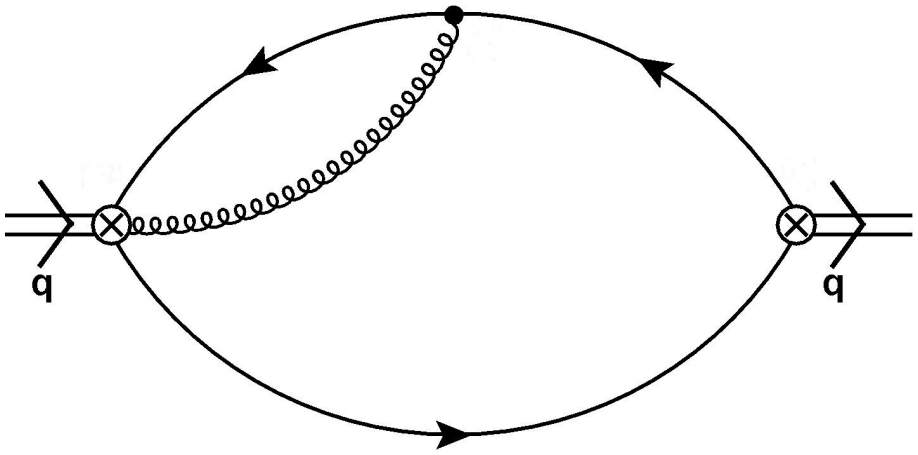

Figure 2.1: Counterclockwise perturbative diagram, representing the (2.12) contraction scheme.

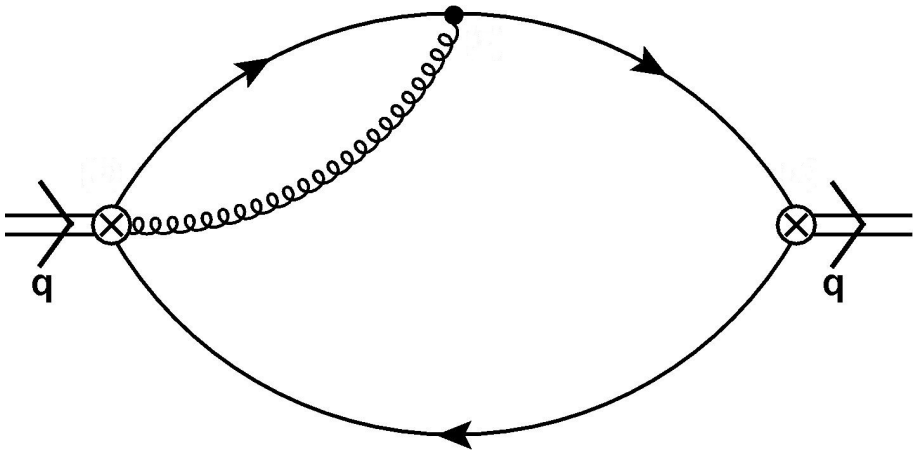

Figure 2.2: Clockwise perturbative diagram, representing the (2.13) contraction scheme.

Note that each of these diagrams is distinct, and in principle both would need to be calculated and the results summed to get the full perturbative contribution. However, using the software we have developed to handle these types of calculations, we were able to show that these diagrams are in fact equal, and thus calculating any one of them and doubling that result would give us the full perturbative contribution. As such, from this point forward we will only be discussing the calculation of the diagram in Figure 2.1 with the understanding that once the calculation is complete, we will simply double our result to get the full perturbative contribution.

Let us continue with the calculation and take a look at the individual contractions.

2.1.2 Field Contractions

Note that in applying the upcoming contractions we are free to commute bosonic fields past each other, but when dealing with fermionic fields each time we commute fields past each other, the term picks up an overall minus sign.

Looking first at the quark field contractions we have:

$$\begin{aligned} \overbrace{Q_i^\alpha(x)\overline{Q}_j^\beta}(y) &= i\delta^{\alpha\beta} S_{ij}(x-y) \\ &= i\delta^{\alpha\beta} \int \frac{d^dp}{(2\pi)^d}\, e^{-ip\cdot(x-y)} S_{ij}(\not{p}) \end{aligned} \tag{2.14}$$

where, in the second line, we have just Fourier transformed our position space propagator $S(x-y)$ into a momentum space propagator $S(\not{p})$ which is defined below in equation (2.19). Here, p is some internal four momentum. Note also that diagrammatically this contraction (2.14) represents a directed fermion line running from y to x.

Our gluon field contractions will then take the following form:

$$\begin{aligned} \overbrace{A_\mu^a(x)A_\nu^b}(y) &= -i\delta^{ab} \int \frac{d^dk}{(2\pi)^d}\, e^{-ik\cdot(x-y)} \Big(\frac{1}{k\cdot k}\big[g_{\mu\nu} - (1-a)\frac{k_\mu k_\nu}{k\cdot k}\big] \Big) \\ &\to -i\delta^{ab} \int \frac{d^dk}{(2\pi)^d}\, e^{-ik\cdot(x-y)} \frac{g_{\mu\nu}}{k\cdot k} \end{aligned} \tag{2.15}$$

where in the second line of (2.15) we have made the choice to work in the Feynman gauge where $a \to 1$. Note that in (2.15), k is some internal four momentum. We will also need this expression with a position space derivative applied to it which is given by

$$\frac{d}{dx^\omega}\overbrace{A_\mu^a(x)A_\nu^b}(y) = -\,\delta^{ab} \int \frac{d^dk}{(2\pi)^d}\, k_\omega e^{-ik\cdot(x-y)} \frac{g_{\mu\nu}}{k\cdot k} \tag{2.16}$$

again in the Feynman gauge.

If we now apply (2.14) and (2.16) to (2.12) while carefully keeping track of the minus signs as we get the quark fields in the necessary positions to apply (2.14), we can rewrite (2.12) in terms of our internal momenta. We can then substitute this result back into (2.5), and evaluate the x and z integrals to turn our exponentials into Dirac delta functions as we know that $\int d^d x \; e^{ik\cdot x} = (2\pi)^d \delta^{(d)}(k)$. Having done so, we get an expression for the perturbative contribution to our correlation function that can be written as follows:

$$\Pi^{\mu\nu}_{(pert)}(q) = \frac{ig_s^2}{4(2\pi)^{2d}} \delta^{ab}\delta^{\alpha\gamma}\delta^{\alpha\xi}\delta^{\delta\epsilon}\gamma^{\mu}_{ij}\,\gamma^{\rho}_{ks}\,\gamma^{5}_{sm}\,\gamma^{\sigma}_{nr}\,\lambda^{a}_{\gamma\delta}\,\lambda^{b}_{\epsilon\xi}\,\epsilon^{\nu\rho\omega\zeta}\Bigg[\int d^dk_1 \int d^dp_1 \int d^dp_2 \int d^dp_3 \left(\frac{g_{\sigma\zeta}k_{1\omega}}{k_1\cdot k_1}\delta(k_1+p_2-p_3)\delta(q+p_3-p_1) S_{ri}(\not{p}_3) S_{jk}(\not{p}_1) S_{mn}(\not{p}_2) \right) - - \int d^dk_2 \int d^dp_1 \int d^dp_2 \int d^dp_3 \left(\frac{g_{\sigma\omega}k_{2\zeta}}{k_2\cdot k_2}\delta(k_2+p_2-p_3)\delta(q+p_3-p_1) S_{ri}(\not{p}_3) S_{jk}(\not{p}_1) S_{mn}(\not{p}_2) \right) \Bigg]. \tag{2.17}$$

This expression can be cleaned up quite a bit if we now evaluate the integrals in p_1 and k_1. Then we apply the identities found in Appendix A Section A.1 to do the color algebra where we find that $\delta^{ab}\delta^{\alpha\gamma}\delta^{\alpha\xi}\delta^{\delta\epsilon}\lambda^{a}_{\gamma\delta}\,\lambda^{b}_{\epsilon\xi} = 16$. Finally, grouping the appropriate terms in a trace gives us:

$$\Pi^{\mu\nu}_{(pert)}(q) = \frac{4ig_s^2}{(2\pi)^{2d}}\epsilon^{\nu\rho\omega\zeta}\int d^dp_2 \int d^dp_3 \left(\frac{g_{\sigma\zeta}(p_{3\omega}-p_{2\omega}) - g_{\sigma\omega}(p_{3\zeta}-p_{2\zeta})}{(p_3-p_2)\cdot(p_3-p_2)} \times \times \mathrm{Tr}\left[\gamma^{\mu}\, S(\not{p}_3+\not{q})\,\gamma^{\rho}\,\gamma^{5}\, S(\not{p}_2)\,\gamma^{\sigma}\, S(\not{p}_3) \right] \right). \tag{2.18}$$

We are now going to want to expand out our momentum space propagators so that we can evaluate the traces in this expression.

2.1.3 Momentum Space Propagators

Our momentum space propagators for a massive particle are defined as:

$$\begin{aligned} S(\not{p}) &= \Big(\frac{\not{p}+m}{p\cdot p-m^2}\Big) \\ S(\not{p}+\not{q}) &= \Big(\frac{\not{p}+\not{q}+m}{(p+q)\cdot(p+q)-m^2}\Big) \end{aligned} \tag{2.19}$$

where m is the mass of the charm quark in this calculation. Using the definitions in (2.19) on the propagators in (2.18) and simplifying gives us:

$$\begin{aligned} \Pi^{\mu\nu}_{(pert)}(q) = \frac{4ig_s^2}{(2\pi)^{2d}}\epsilon^{\nu\rho\omega\zeta}\int d^dp_2\int d^dp_3\Bigg(&\big(g_{\sigma\zeta}(p_{3\omega}-p_{2\omega})-g_{\sigma\omega}(p_{3\zeta}-p_{2\zeta})\big)\times \\ &\times\frac{\mathrm{Tr}\big[\gamma^\mu\,(\not{p}_3+\not{q}+m)\,\gamma^\rho\,\gamma^5\,(\not{p}_2+m)\,\gamma^\sigma\,(\not{p}_3+m)\big]}{(p_3-p_2)\cdot(p_3-p_2)(p_3\cdot p_3-m^2)(p_2\cdot p_2-m^2)\big((p_3+q)\cdot(p_3+q)-m^2\big)}\Bigg). \end{aligned} \tag{2.20}$$

2.1.4 Traces and Dirac Algebra

At this point, we are ready to evaluate the trace in equation (2.20). In practice, this trace would be expanded into twelve traces and we would then evaluate each trace individually. We would do so by commuting the γ^5 to the right, and using the identities in Appendix A Section A.2 to evaluate the traces. In the end this trace would turn into 34 distinct terms involving four-momenta, our mass m, Levi-Civita symbols and metric tensors.

Note that we use a convention for γ^5 in d dimensions where γ^5 can be written as follows:

$$\gamma^5 = -\frac{i}{24}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho. \tag{2.21}$$

This convention is consistent with [23]. The identities defined in Appendix A Section A.2

are also based on this convention if we are working in d dimensions.

At this point, it is no longer useful to write out the expression in full due to its size, but we can briefly discuss some of the steps needed to continue simplifying our correlator. In practice, all of these manipulations were done on a computer using the Mathematica package we have developed to handle this kind of work (much of this code is included in Appendix B).

The next step would be to multiply our 34 terms from the trace into the 4 factors made up of metric tensors we have inside our double integral, resulting in some 136 terms. Each of these terms would have a Levi-Civita symbol in it along with some mix of masses, momenta and metrics. Each of these Levi-Civita symbols could then be contracted with our pre-factor Levi-Civita symbol using the identity listed in Appendix A Section A.2. At this point, our expression will have blown up to about 2000 terms but it will be ready for a lot of simplification. Each term will have a number of Lorenz indices that can be contracted. Once all contractions have been done, we will be left with only a little over a hundred terms, each consisting of factors of d, m, momenta and/or a metric tensor. Each of these terms must have exactly two remaining Lorenz indices which must be μ and ν.

2.1.5 Projection Operators

We now apply projection operators to extract the scalar $J = 0$ and vector $J = 1$ parts of our correlator.

Our correlator could be written as follows:

$$\Pi_{\mu\nu}(q) = \frac{q_\mu q_\nu}{q^2}\Pi_0(q) + \Big(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\Big)\Pi_1(q) \tag{2.22}$$

where $\Pi_0(q)$ and $\Pi_1(q)$ represent the scalar and vector portions of the correlator respectively [8]. This means that we can write

$$\Pi_0(q) = \frac{q_\mu q_\nu}{q^2}\Pi^{\mu\nu}(q) \tag{2.23}$$

and

$$\Pi_1(q) = \Big(\frac{1}{d-1}\Big)\Big(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\Big)\Pi^{\mu\nu}(q). \tag{2.24}$$

As such we will refer to $\big(\frac{q_\mu q_\nu}{q^2}\big)$ as the scalar projection operator and $\big(\frac{1}{d-1}\big)\big(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\big)$ as the vector projection operator. If we now apply the scalar and vector projectors to our specific correlator, we get two expressions, one for $\Pi_0^{(pert)}(q)$ and another for $\Pi_1^{(pert)}(q)$.

Note that after fully simplifying $\Pi_0^{(pert)}(q)$ with the steps described in the next few sections, we expect to find that $\Pi_0^{(pert)}(q)$ will turn out to be zero; the reason being that $j^\mu_{(m)}(x)$ (2.3) is a conserved current. As such, hitting both sides of (2.22) with $\big(\frac{q_\mu q_\nu}{q^2}\big)$ then rewriting these external momenta as derivatives of the exponential factor in (2.1) and integrating by parts to shift the derivatives, shows us that $\Pi_0 \sim \partial_\mu j^\mu_{(m)} = 0$. As a double check of our software, we ran the calculation on the $\Pi_0^{(pert)}(q)$ portion to completion and found that it did in fact go to zero. On the other hand, we expect a non-zero result for $\Pi_1^{(pert)}(q)$. In the end we will find that $\Pi_1^{(pert)}(q)$ is non-zero as expected.

At this point, we will apply the vector projection operator to the simplified version of $\Pi^{\mu\nu}_{(pert)}(q)$ in (2.20) to give us $\Pi_1^{(pert)}(q)$. This $\Pi_1^{(pert)}(q)$, which is the perturbative contribution to the vector portion of our cross-correlator, will now be free of any Lorenz indices. It will simply be phrased in terms of two internal momenta to be integrated over, the external momentum and other factors including m, d, g_s and other constants. In effect, what we have done at this stage is reduced the problem of calculating the perturbative vector contribution of our original $\Pi^{\mu\nu}(q)$ (2.5), to the problem of solving the hundred or so distinct but similar double integrals that make up our $\Pi_1^{(pert)}(q)$. In principle, this would still be a very complicated problem but Tarasov [24, 25] provides us with a solution to this problem and TARCER [12] provides us with a software package to implement this solution.

2.1.6 TARCER

TARCER [12] is a Mathematica package which reduces two-loop propagator integrals to some combination of master integrals. Some of these master integrals have known solutions and in our case they are all known. TARCER implements the reduction algorithm developed by Tarasov [24, 25] and expands on some of the recurrence relations detailed in these papers. TARCER is widely used to solve integrals of this sort and is exactly what we need to solve the integrals in our $\Pi_1^{(pert)}(q)$.

With q as our external momentum and m our mass along with internal momenta k_1 and k_2 and abbreviations $(k_3 = k_1 - q)$, $(k_4 = k_2 - q)$ and $(k_5 = k_1 - k_2)$. Assuming that u, v, r, s, t and a, b, c, d, e are all nonnegative integers, TARCER will evaluate integrals of the following form when translated to its TFI format as follows:

$$\begin{aligned}&\frac{1}{\pi^d}\iint \frac{d^dk_1 d^dk_2 (k_1^2)^u (k_2^2)^v (qk_1)^r (qk_1)^s (k_1k_2)^t}{[k_1^2 - m_1^2]^a [k_2^2 - m_2^2]^b [k_3^2 - m_3^2]^c [k_4^2 - m_4^2]^d [k_5^2 - m_5^2]^e} \\ &= TFI\big[d, q^2, \{u, v, r, s, t\}, \{\{a, m_1\}, \{b, m_2\}, \{c, m_3\}, \{d, m_4\}, \{e, m_5\}\}\big].\end{aligned} \tag{2.25}$$

Note that TFI will accept more optional arguments to reduce an even wider variety of integrals but for our purposes, this is the form of the TFI that will be used. It will now be useful to define

$$z = \frac{q \cdot q}{4m^2}. \tag{2.26}$$

Also, note that ${}_pF_q\left(a_1, \cdots, a_p; b_1, \cdots, b_q; z\right)$ indicates a hypergeometric function [26] defined as:

$${}_pF_q\left(a_1, \cdots, a_p; b_1, \cdots, b_q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{2.27}$$

where the $(a)_n$ and $(b)_n$ are Pochhammer symbols defined by:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.28}$$

and the Γs are gamma functions. Using this notation TARCER will return integrals of the forms TAI, TBI and TJI among others. The integral outputs generated by TARCER for this calculation all have solutions listed in (2.29), (2.30) and (2.31).

$$\begin{aligned}\mathrm{TAI}\big[d, q^2, \{\{a, m\}\}\big] &= \frac{1}{\pi^{d/2}} \int \frac{d^d k_1}{[k_1^2 - m^2]^a} \\ &= i^{(1-d)} (-m^2)^{(d/2-a)} \frac{\Gamma[a - \frac{d}{2}]}{\Gamma[a]}\end{aligned} \tag{2.29}$$

$$\begin{aligned}&\mathrm{TBI}\big[d, q^2, \{\{a, m\}, \{b, m\}\}\big] \\ &\quad = \frac{1}{\pi^{d/2}} \int \frac{d^d k_1}{[k_1^2 - m^2]^a [k_3^2 - m^2]^b} \\ &\quad = i^{(1-d)} (-m^2)^{(d/2-a-b)} \frac{\Gamma[a + b - \frac{d}{2}]}{\Gamma[a+b]} \times \\ &\qquad \times \ {}_3F_2\left[a, b, a + b - \frac{d}{2}; \frac{(a+b)}{2}, \frac{(a+b+1)}{2}; z\right]\end{aligned} \tag{2.30}$$

$$\begin{aligned}&\mathrm{TJI}\big[d, q^2, \{\{a, m\}, \{b, m\}, \{c, 0\}\}\big] \\ &= \frac{1}{\pi^d} \iint \frac{d^d k_1 d^d k_2}{[k_1^2 - m^2]^a [k_5^2 - m^2]^b [k_4^2]^c} \\ &= (-1)^{(1-d)} (-m^2)^{(d-a-b-c)} \frac{\Gamma[a+b+c-d]\Gamma[\frac{d}{2} - c]\Gamma[b + c - \frac{d}{2}]\Gamma[a + c - \frac{d}{2}]}{\Gamma[a]\Gamma[b]\Gamma[\frac{d}{2}]\Gamma[a+b+2c-d]} \times \\ &\quad \times \ {}_4F_3\left[b, a+b+c-d, b+c-\frac{d}{2}, a+c-\frac{d}{2}; \frac{d}{2}, c+\frac{(a+b-d)}{2}, c+\frac{(a+b-d+1)}{2}; z\right]\end{aligned} \tag{2.31}$$

Note that the expression (2.29) can be derived easily from equations in [10], the expression (2.30) is found in [27] and the expression (2.31) is found in [28].We should also note that there are other possible variations of outputs from TARCER but, for this calculation, we only encounter the three we have shown here.

We will now take our $\Pi_1^{(pert)}(q)$ and, after some formatting, pass it to TARCER. The TARCER output will be in the form of TAIs, TBIs and/or TJIs. We will substitute in the known solutions to these listed in (2.29), (2.30) and (2.31). Finally after some simplification we will have the exact leading order result for the perturbative vector contribution from our correlation function. Again see Appendix B for much of the code implementing these steps.

2.1.7 Exact Leading Order Perturbative Result

At leading order, the exact d-dimensional perturbative vector contribution to our correlator is given by:

$$\begin{aligned}\Pi_1^{(pert)}(q) = 2\frac{4^{2-d}(d-3)(d-2)e^{-i\pi d}\pi^{-d}g_s^2\left(-m^2\right)^d}{3(d-1)m^4\Gamma(4-d)\Gamma(5-d)\Gamma\left(\frac{d}{2}\right)}\Bigg(\\ -2(d-2)\Gamma(4-d)\Gamma(5-d)\Gamma\left(1-\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}\right) \\ -3(2(d-2)z+1)\Gamma(4-d)\Gamma(5-d)\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\times \\ \times\ {}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};z\right) \\ +(2(d-2)z+d-3)\Gamma(3-d)\Gamma(5-d)\Gamma\left(2-\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}-1\right)\times \\ \times\ {}_3F_2\left(1,3-d,2-\frac{d}{2};\frac{5}{2}-\frac{d}{2},\frac{d}{2};z\right) \\ -4(z-1)\Gamma(4-d)^2\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(3-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)\times \\ \times\ {}_3F_2\left(1,4-d,2-\frac{d}{2};\frac{5}{2}-\frac{d}{2},\frac{d}{2};z\right)\Bigg).\end{aligned} \tag{2.32}$$

It should be noted that in this final step, we have also included the factor of 2 needed to account for the other direction in which we could have contracted our fields.

2.1.8 Epsilon Expansion and Result

Now that all integrals have been evaluated and we have our exact leading order result, we can proceed with the next step in the implementation of dim-reg. We will now want to let our spacetime dimension $d \to 4 + 2\epsilon$ as we mentioned in Chapter 1. We will then want to expand this result about $\epsilon = 0$. To do so, we will need to be able to expand hypergeometric functions in ϵ where the Pochhammer symbol constituents of the hypergeometric function have ϵ dependence. We found that the easiest way to do this expansion was to use the HypExp package for Mathematica. HypExp is a widely used Mathematica package designed to expand hypergeometric functions. More details about HypExp and the other Mathematica package it uses, HPL, are available in references [29, 30, 31, 32].

Before we can proceed with this expansion, we have one more detail to discuss. In implementing dim-reg we glossed over a subtle but important fact. In four dimensions, our coupling g_s is dimensionless but in the transition to d dimensions g_s becomes dimensionful. We then make g_s dimensionless again by off loading its dimensionality into a dimensionful renormalization scale parameter that we introduce, labelled ν. We do this such that when using the convention $d \to 4 + 2\epsilon$, we have $g_s \to g_s \nu^{\epsilon}$. We account for this adjustment by introducing a pre-factor to our correlator just before we do the ϵ-expansion. In our final answer, we will find that we only have factors of ν appearing in conjunction with mass terms such that we get terms that look like $\log(-\frac{m^2}{\nu^2})$. This is ideal and expected as ν has dimensions of mass and as such it renders these terms dimensionally correct.

We now separate out the factors of the hypergeometric functions and the terms with no hypergeometric functions in them. We expand these terms in Laurent series about $\epsilon = 0$ to first order in ϵ. Picking off the lowest order term in ϵ we expand our hypergeometrics out to the appropriate order using HypExp and put all the terms back together. Finally we drop all terms that go like positive powers of ϵ as they will eventually go to zero and we are left with our final ϵ-expanded result.

The ϵ-expanded perturbative vector contribution to our correlator is given by:

$$
\begin{aligned}
\Pi^{(pert)}_{1(\epsilon)}(q) = 2\Bigg(& \frac{g_s^2 m^4 z}{6\pi^4\epsilon^2} - \\
& - \frac{g_s^2 m^4}{36\pi^4\epsilon A(z)}\Big((-12z\log(M) + 4z^2 + 12i\pi z + z + 6)A(z) + \\
& \quad + 3i(4z+1)B(z)\Big) + \\
& + \frac{g_s^2 m^4}{432\pi^4 z A(z)}\Big(9\left(8z^2 - 4z + 1\right)A(z)B^2(z) + \\
& \quad + 12izB(z)\left(-6(4z+1)\log(M) + 8z^2 + 6z + 6i\pi(4z+1) - 1\right) + \\
& \quad + 2zA(z)\left(76z^2 + 12i\pi\left(4z^2 + z + 6\right) - 66\pi^2 z + 61z + 6\right) + \\
& \quad + 24zA(z)\left(6z\log^2(M) - \left(4z^2 + 12i\pi z + z + 6\right)\log(M)\right) + \\
& \quad + 18iz(4z+1)\Big(\big(\log(1 + iA(z)) + \log(\frac{1}{4}(1 - iA(z)))\big)\times \\
& \qquad \times \big(\log(1 + iA(z)) - \log(1 - iA(z))\big)\Big) + \\
& \quad + 36iz(4z+1)\Big(\mathrm{Li}_2\big(\frac{1}{2}(iA(z)+1)\big) - \mathrm{Li}_2\big(\frac{1}{2}(1 - iA(z))\big)\Big)\Big)\Bigg)
\end{aligned}
\tag{2.33}
$$

where

$$
\begin{aligned}
A(z) &= \sqrt{\frac{z}{1-z}} \\
B(z) &= \log\left(\frac{i - A(z)}{A(z) + i}\right) \\
M &= -\frac{m^2}{\nu^2}.
\end{aligned}
\tag{2.34}
$$

It should be noted that in (2.33) the $\mathrm{Li}_s(z)$ functions represent polylogarithms [26] which are given by:

$$
\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{2.35}
$$

Also note that (2.33) contains a non-polynomial divergence that we will be discussing further in Chapter 3.

2.2 Non-perturbative Contributions

We start our discussion about the non-perturbative contributions to our correlator by looking back at equation (2.5). At the beginning of our discussion about the perturbative contribution, we pulled the TOP out of this expression and split it up into two TOPs. We then went on to apply Wick's theorem to fully contract all of our quark and gluon fields. Since at the time, we were looking for only the perturbative contribution we only considered terms where all fields were fully contracted and ignored those with uncontracted fields. But in the discussion about Wick's theorem in Chapter 1, we did mention that these VEVs of uncontracted fields can be non-zero due to the nature of the QCD vacuum. It is now time to consider the contribution to our cross-correlator from these terms containing non-zero VEVs.

All of these contributing terms will contain what we call condensates (a term that we will define in more detail shortly), and so we will refer to these terms as condensate terms. Note also, that we will refer to our condensates as 4D or 6D condensates etc. this language refers to the mass dimension of the respective condensate. Let us now recall the contractions laid out in equations (2.9), (2.10) and the contraction of the gluon field strength tensor with the gluon field emerging from the interaction term. We can see that removing any one of the contractions will leave us with a term containing the VEV of two uncontracted fields and, in principle, in QCD all of these would be non-zero. However since in this calculation we are dealing with only heavy quarks, the contributions from condensates involving quarks can be ignored. Nonetheless, we are still left with a term in which just the gluonic fields are left uncontracted and this term will contribute.

To begin this non-perturbative calculation we start again from equation (2.5). We will follow the same steps outlined in the beginning of our perturbative calculation, first applying Wick's theorem to the time ordered product of our quark fields only, this time, leaving the gluon fields

untouched. We then perform the needed contractions on our quark fields. After simplifying some of the algebra and rearranging terms we are left with:

$$\Pi^{\mu\nu}(q) = -\frac{i}{8(2\pi)^{3d}}\,\gamma^{\mu}_{ij}\,\gamma^{\rho}_{ks}\,\gamma^{5}_{sm}\,\gamma^{\sigma}_{nr}\,\epsilon^{\nu\rho\omega\zeta}\int d^dx\int d^dz\int d^dp_1\int d^dp_2\int d^dp_3\Bigg(\\ e^{iz\cdot(p_2-p_3)}e^{ix\cdot(q+p_3-p_1)}S_{ri}(\not{p}_3)S_{jk}(\not{p}_1)S_{mn}(\not{p}_2)\Big[g_s^2\ \mathrm{Tr}[\lambda^a\lambda^b]\big\langle{:}\,G^a_{\omega\zeta}(0)\,A^b_{\sigma}(z){:}\big\rangle\Big]\Bigg). \tag{2.36}$$

We will now take a closer look at the square bracketed expression in (2.36) which contains our non-local VEV and a few pre-factors that will help us evaluate it.

2.2.1 Expansion of the Non-local VEV

We now need to evaluate

$$g_s^2\ \mathrm{Tr}[\lambda^a\lambda^b]\big\langle{:}\,G^a_{\omega\zeta}(0)\,A^b_{\sigma}(z){:}\big\rangle. \tag{2.37}$$

We are going to want to start by expanding this non-local VEV into an infinite series of local VEVs. To do that we can use the ideas discussed in [11]. Specifically we know that working in the fixed point gauge where $x^\mu A_\mu(x) = 0$, we can write our gluon field at z as an infinite series of terms involving the commutator of covariant derivatives D and gluon field strength tensors G at the origin as follows:

$$A_\sigma(z) = \sum_{n=0}^{\infty}\frac{1}{n!(n+2)}z^{\phi}z^{\rho_1}z^{\rho_2}\ldots z^{\rho_n}[D_{\rho_1}(0),[D_{\rho_2}(0),[\ldots[D_{\rho_n}(0),G_{\phi\sigma}(0)]\ldots]]]. \tag{2.38}$$

Note that in the perturbative calculation, when contracting gluon fields, we chose to work in the Feynman gauge, however here we have chosen to work in the fixed point gauge. Perhaps

the easiest way to think about why we are free to do this is to realize that for each gauge invariant contribution to our correlator, we are free to make whatever gauge choice we wish. A more detailed discussion on this topic is available in [33]. It will also be useful to note that we can write our gluon field strength tensor $G_{\mu\nu}$ in the following form:

$$G_{\mu\nu} = \frac{ig_s}{2}\lambda^a G^a_{\mu\nu}. \tag{2.39}$$

Also we can write individual gluon fields A_μ similarly as follows:

$$A_\mu = \frac{ig_s}{2}\lambda^a A^a_\mu. \tag{2.40}$$

Now substituting (2.39) and (2.40) into (2.38) then substituting the result into (2.37) will give us:

$$\begin{aligned} g_s^2\ \mathrm{Tr}[\lambda^a\lambda^b]\big\langle :G^a_{\omega\zeta}(0)\,A^b_\sigma(z): \big\rangle &= \\ &= g_s^2\ \mathrm{Tr}[\lambda^a\lambda^b]\Big\{\frac{1}{2}z^\phi\big\langle :G^a_{\omega\zeta}(0)\,G^b_{\phi\sigma}(0): \big\rangle + \frac{1}{3}z^\phi z^{\rho_1}\big\langle :G^a_{\omega\zeta}(0)[D_{\rho_1}(0), G^b_{\phi\sigma}(0)]: \big\rangle \\ &\qquad + \frac{1}{8}z^\phi z^{\rho_1} z^{\rho_2}\big\langle :G^a_{\omega\zeta}(0)[D_{\rho_1}(0),[D_{\rho_2}(0), G^b_{\phi\sigma}(0)]]: \big\rangle + \dots\Big\} \\ &= -2\ z^\phi\ \mathrm{Tr}[\big\langle :G_{\omega\zeta}(0)\,G_{\phi\sigma}(0): \big\rangle] - \frac{4}{3}\ z^\phi z^{\rho_1}\ \mathrm{Tr}[\big\langle :G_{\omega\zeta}(0)[D_{\rho_1}(0), G_{\phi\sigma}(0)]: \big\rangle] \\ &\qquad - \frac{1}{2}\ z^\phi z^{\rho_1} z^{\rho_2}\ \mathrm{Tr}[\big\langle :G_{\omega\zeta}(0)[D_{\rho_1}(0),[D_{\rho_2}(0), G_{\phi\sigma}(0)]]: \big\rangle] - \dots \end{aligned} \tag{2.41}$$

where the second equality holds because of another application of (2.39). The VEV in the first term on the right will give us our term proportional to what we will call the 4D gluon condensate. The second term on the right will go to zero. An easy way to see this is there are an odd number of Lorenz indices in this term and there would be no way to write this term using only metric tensors. The VEV in the third term will give us a term proportional to what we will call the 6D gluon condensate. We truncate the series at this point ignoring higher order terms which will be getting increasingly suppressed by factors of $\frac{1}{m}$ and $\frac{1}{q}$.

2.2.2 Evaluating the 4D VEV

Our 4D VEV can be written in terms of metric tensors as follows:

$$\mathrm{Tr}[\langle : G_{\omega\zeta} G_{\phi\sigma} : \rangle] = A g_{\omega\zeta} g_{\phi\sigma} + B g_{\phi\zeta} g_{\omega\sigma} + C g_{\omega\phi} g_{\sigma\zeta} \tag{2.42}$$

where A, B and C are unknown constants we will need to solve for. But we know that our gluon field strength tensors are antisymmetric under the exchange of their indices so $A \to 0$. Now we can group the remaining two terms while maintaining antisymmetry under $\omega \leftrightarrow \zeta$ and $\phi \leftrightarrow \sigma$ with a new arbitrary constant F as follows:

$$\mathrm{Tr}[\langle : G_{\omega\zeta} G_{\phi\sigma} : \rangle] = F\big[g_{\omega\sigma} g_{\zeta\phi} - g_{\omega\phi} g_{\zeta\sigma}\big]. \tag{2.43}$$

We can now start solving for F by contracting both sides of the equation with $g_{\omega\sigma} g_{\zeta\phi}$ giving us:

$$\mathrm{Tr}[\langle : G_{\omega\phi} G_{\phi\omega} : \rangle] = F\big[d^2 - d\big]. \tag{2.44}$$

Then, by again using (2.39) and evaluating the remaining trace using $\mathrm{Tr}[\lambda^a \lambda^b] = 2\delta^{ab}$ we can solve for F. We find that:

$$F = \frac{\langle g_s^2 G^2 \rangle}{2d(d-1)} \tag{2.45}$$

where

$$\langle g_s^2 G^2 \rangle = g_s^2 \langle : G^a_{\omega\phi} G^a_{\omega\phi} : \rangle \tag{2.46}$$

is our 4D gluon condensate. We can now substitute (2.45) back into (2.43) and using (2.39)

rewrite the left hand side (LHS) of the expression. If we then substitute this back into the first term in (2.41) and keeping just this first term, substitute the result back into (2.36), we have an expression for our 4D condensate contribution.

2.2.3 Evaluating the 4D Contribution

We can write the contribution from the 4D condensate term as follows:

$$\Pi^{\mu\nu}_{(4D)}(q) = \frac{i}{8(2\pi)^{3d}}\,\gamma^{\mu}_{ij}\,\gamma^{\rho}_{ks}\,\gamma^{5}_{sm}\,\gamma^{\sigma}_{nr}\,\epsilon^{\nu\rho\omega\zeta}\int d^dx\int d^dz\int d^dp_1\int d^dp_2\int d^dp_3\Bigg(\\ e^{iz\cdot(p_2-p_3)}e^{ix\cdot(q+p_3-p_1)}S_{ri}(\not{p}_3)S_{jk}(\not{p}_1)S_{mn}(\not{p}_2)\Big[z^{\phi}\frac{\langle g_s^2G^2\rangle}{d(d-1)}\big[g_{\omega\sigma}g_{\zeta\phi}-g_{\omega\phi}g_{\zeta\sigma}\big]\Big]\Bigg). \tag{2.47}$$

This expression corresponds to the Feynman diagram represented in Figure 2.3. Again we found that the clockwise direction contributes equally to the calculation and so we will need to remember to include an overall factor of 2 when we write our final contribution from the 4D term.

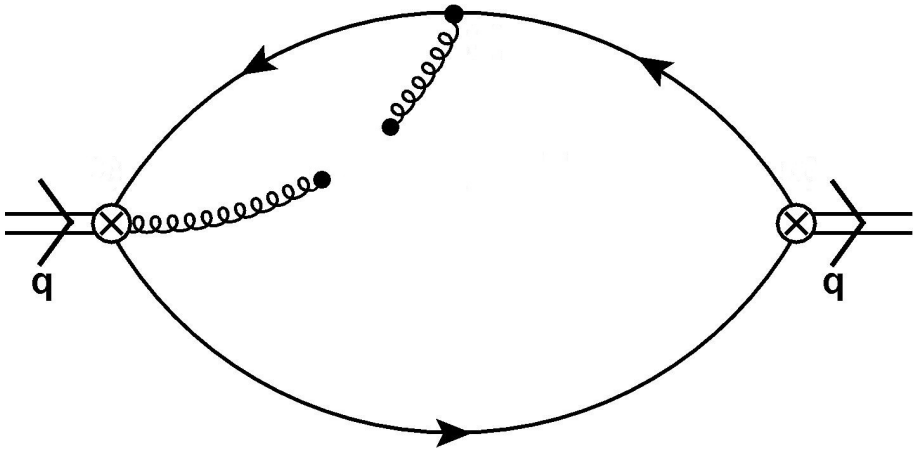

Figure 2.3: Counterclockwise non-perturbative 4D gluon condensate diagram, representing the contribution from equation (2.47).

To simplify (2.47), we will now want to write the z^{ϕ} term as the derivative of the exponential term in z as $z^{\phi}e^{iz\cdot(p_2-p_3)} = -i\frac{d}{dp_2^{\phi}}e^{iz\cdot(p_2-p_3)}$. Then by evaluating the appropriate integral ($\int d^dp_2$

in this case), we can use integration by parts to shift this derivative onto the appropriate propagator ($S_{mn}(\not{p}_2)$ in this case) at the cost of a surface term that vanishes in dim-reg. Then, evaluating our position space integrals giving us delta functions followed by evaluating the appropriate momentum-space integrals and organising the appropriate terms into a trace leaves us with:

$$\Pi^{\mu\nu}_{(4D)}(q) = -\frac{\left\langle g_s^2 G^2 \right\rangle}{8d(d-1)(2\pi)^d}\epsilon^{\nu\rho\omega\zeta}\int d^d p_3 \Bigg(\\ \left(g_{\omega\sigma}g_{\zeta\phi} - g_{\omega\phi}g_{\zeta\sigma}\right)\mathrm{Tr}\left[\gamma^\mu\, S(\not{p}_3 + \not{q})\,\gamma^\rho\,\gamma^5\,\left(\frac{\partial}{\partial p_3^\phi}\right)S(\not{p}_3)\,\gamma^\sigma\, S(\not{p}_3)\right]\Bigg). \tag{2.48}$$

Notice the similarities in the structure of equations (2.48) and (2.18). One of the key differences between these two expressions is that in the 4D case, one of our momentum-space propagators has a derivative acting on it. This derivative can be taken in one of two ways, the first way would be to apply the identity $\left(\frac{\partial}{\partial p_\phi}\right)S(\not{p}) = S(\not{p})\gamma^\phi S(\not{p})$ which would result in a trace with more gamma matrices. The other approach would be to expand the propagator, using (2.19), and apply the derivative directly to the internal momentum using simple identities like $\left(\frac{\partial}{\partial p_\phi}\right)p^\mu = g^{\phi\mu}$ and $\left(\frac{\partial}{\partial p_\phi}\right)p^2 = 2p^\phi$. We used both techniques as a double check and found that the results agreed.

Once the derivative is handled, the calculation closely mimics that of the perturbative portion. We expand, deal with the traces and contract the appropriate products of metrics and Levi-Civita symbols. Once done, we are again left with a large number of terms (about a hundred or so) involving factors of d, m, momenta and/or a metric tensor. Each of these terms again must have exactly two remaining Lorenz indices which must be μ and ν.

To continue the calculation we again project out the scalar and vector portions of our correlator. We find, as expected, that the scalar portion is again zero and we proceed with the vector portion. In this case, our expression has only one integral and, as such, we need to artificially introduce a second integral to be able to use the Tarasov recurrence relations. So

we multiply by what is called a massive tadpole which is, in essence, multiplying by a simple massive integral with a known solution. Once in this form, the integrals are again ready to have the Tarasov recurrence relations applied to them and after TARCER is done, we divide out the massive tadpole. In the language of TARCER, we are in effect dividing out a TAI.

At this point, the expression is written in terms of the master integrals with solutions listed in (2.29), (2.30) and (2.31). After substituting in these solutions and simplifying we are left with our result.

2.2.4 Exact Leading Order 4D Gluon Condensate Result

The exact leading order d-dimensional 4D gluon condensate vector contribution to our correlator is given by:

$$\Pi_1^{(4D)}(q) = 2\left\langle g_s^2 G^2\right\rangle \frac{(2i)^{-d}(d-3)(d-2)^2\pi^{-\frac{d}{2}} z\left(-m^2\right)^{d/2}\Gamma\left(3-\frac{d}{2}\right)}{d(d-1)m^4(z-1)} \times \\ \times\ {}_2F_1\left(1, 2-\frac{d}{2}; \frac{3}{2}; z\right). \tag{2.49}$$

Note that here we have again added an extra factor of 2 to account for the clockwise diagram.

2.2.5 Epsilon Expanded 4D Gluon Condensate Result

We will again want to ϵ-expand our exact leading order result. To do so we follow exactly the same steps and procedures outlined in the perturbative case. Once everything has been expanded and simplified we find that the ϵ-expanded 4D gluon condensate vector contribution to our correlator is given by

$$\Pi^{(4D)}_{1(\epsilon)}(q) = \left\langle g_s^2 G^2 \right\rangle \frac{2z + iA(z)B(z)}{48\pi^2 z}. \tag{2.50}$$

2.2.6 Evaluating the 6D VEV

Now returning back to equation (2.41) we have one more term in our series that will contribute. To get at the 6D gluon condensate contribution we will need to simplify the trace in the last term in (2.41). We could again write out all of the combinations of metrics that this 6D term could depend on like we did in the 4D case in equation (2.42). If we did, we would find 15 possible combinations of metrics. After taking into account the antisymmetric nature of the gluon field strength tensors, we could immediately eliminate five of these leaving us with ten terms. We could then start to carefully group these terms so that the symmetries of the gluon field strength tensors are preserved until we found that this term could be written as follows:

$$\begin{aligned}
\mathrm{Tr}[\langle : G_{\omega\zeta}(0)[D_{\rho_1}(0),&[D_{\rho_2}(0), G_{\phi\sigma}(0)]]: \rangle] = \\
&= A\, g_{\rho_1\rho_2}(g_{\omega\phi}g_{\zeta\sigma} - g_{\phi\zeta}g_{\omega\sigma}) + \\
&\quad + B\, \left[g_{\phi\rho_2}(g_{\omega\rho_1}g_{\zeta\sigma} - g_{\rho_1\zeta}g_{\omega\sigma}) - g_{\sigma\rho_2}(g_{\phi\zeta}g_{\omega\rho_1} - g_{\omega\phi}g_{\zeta\rho_1})\right] + \\
&\quad + C\, \left[g_{\rho_1\sigma}(g_{\omega\rho_2}g_{\phi\zeta} - g_{\omega\phi}g_{\zeta\rho_2}) - g_{\rho_1\phi}(g_{\omega\rho_2}g_{\zeta\sigma} - g_{\omega\sigma}g_{\rho_2\zeta})\right]
\end{aligned} \tag{2.51}$$

where A, B and C are unknown constants for which we will need to solve. We will now need to generate three equations to solve for our three unknowns. The easiest way to do this is to select three different contraction schemes that will fully contract the right hand side (RHS) of the equation without sending it to zero. We will then apply these contraction schemes to both sides of the equation to generate the three equations in three unknowns. We have selected the following three terms to contract both sides:

$$\begin{aligned} &g_{\rho_1\rho_2}g_{\omega\phi}g_{\zeta\sigma} \qquad (1) \\ &g_{\phi\rho_2}g_{\omega\rho_1}g_{\zeta\sigma} \qquad (2) \\ &g_{\rho_1\sigma}g_{\omega\rho_2}g_{\phi\zeta} \qquad (3). \end{aligned} \tag{2.52}$$

The simplification of the RHSs of our three new expressions is fairly straightforward. We simply contract the indices and make use of the fact that $g_{\mu\mu} = d$. The simplification of the LHSs will be a bit more involved. To simplify the LHSs we will need to make use of the Jacobi identity [11] which in this context can be stated as follows:

$$[D_\mu, [D_\nu, D_\rho]] + [D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] = 0. \tag{2.53}$$

We will also make use of the definition $[D_\mu, D_\nu] = -G_{\mu\nu}$ and that $[G_{\mu\nu}, D_\mu] \to 0$ at leading order. We will now take a look at how the simplification of the LHS would work for our contraction scheme (1) found in (2.52). The other two contractions would follow similarly. Applying the metrics in contraction scheme (1) to the LHS of (2.51) and making use of (2.53) while suppressing the arguments and the trace allows us to write

$$\big\langle : G_{\omega\zeta}[D_{\rho_1}, [D_{\rho_1}, G_{\omega\zeta}]] : \big\rangle = -\big\langle : G_{\omega\zeta}[D_{\rho_1}, [D_\zeta, G_{\rho_1\omega}]] : \big\rangle - \big\langle : G_{\omega\zeta}[D_{\rho_1}, [D_\omega, G_{\zeta\rho_1}]] : \big\rangle. \tag{2.54}$$

We can also use (2.53) to show that

$$[D_{\rho_1}, [D_\zeta, G_{\rho_1\omega}]] = [G_{\rho_1\omega}, G_{\rho_1\zeta}] \;\text{ and }\; [D_{\rho_1}, [D_\omega, G_{\zeta\rho_1}]] = [G_{\zeta\rho_1}, G_{\rho_1\omega}]. \tag{2.55}$$

Substituting these results back into (2.54) and flipping commutators in one factor and indices in another to pick up some minus signs gives us

$$\begin{aligned}\left\langle : G_{\omega\zeta}[D_{\rho_1}, [D_{\rho_1}, G_{\omega\zeta}]] : \right\rangle &= \left\langle : G_{\omega\zeta}[G_{\rho_1\zeta}, G_{\rho_1\omega}] : \right\rangle + \left\langle : G_{\omega\zeta}[G_{\rho_1\zeta}, G_{\rho_1\omega}] : \right\rangle \\ &= -4\left\langle : G_{\omega\zeta}\ G_{\zeta\rho_1} G_{\rho_1\omega} : \right\rangle\end{aligned} \tag{2.56}$$

where the second equality is realized by expanding the commutators, rearranging indices and grouping terms. Now wrapping both sides of the equation in a trace (as we must to regain (2.51)) and making use of (2.39) allows us to write

$$\mathrm{Tr}\left[\left\langle : G_{\omega\zeta}[D_{\rho_1}, [D_{\rho_1}, G_{\omega\zeta}]] : \right\rangle\right] = -4\left(\frac{ig_s}{2}\right)^3 \mathrm{Tr}[\lambda^a\lambda^b\lambda^c]\left\langle : G^a_{\omega\zeta}\ G^b_{\zeta\rho_1} G^c_{\rho_1\omega} : \right\rangle. \tag{2.57}$$

The identity $\mathrm{Tr}[\lambda^a\lambda^b\lambda^c] = 2(d_{abc} + if_{abc})$, which can be found in [11], where the d_{abc} is real and totally symmetric and the f_{abc} is again a totally antisymmetric structure constants, can be used to simplify this expression. Then using the totally symmetric nature of d_{abc} and the fact that $G^a_{\mu\nu}$ is antisymmetric in μ and ν to argue away the first term we can write:

$$\mathrm{Tr}\left[\left\langle : G_{\omega\zeta}[D_{\rho_1}, [D_{\rho_1}, G_{\omega\zeta}]] : \right\rangle\right] = -\left\langle g_s^3 G^3 \right\rangle \tag{2.58}$$

where

$$\left\langle g_s^3 G^3 \right\rangle = g_s^3 f_{abc}\left\langle : G^a_{\omega\zeta}\ G^b_{\zeta\rho_1} G^c_{\rho_1\omega} : \right\rangle. \tag{2.59}$$

We can now perform similar simplifications using all three of our contraction schemes in (2.52) on (2.51). Doing so generates the following system of equations:

$$\begin{aligned} d(d-1)(dA + 2B - 2C) &= -\left\langle g_s^3 G^3 \right\rangle \\ d(d-1)(A + (d-1)B - C) &= 0 \\ -d(d-1)(A + B - (d-1)C) &= \frac{\left\langle g_s^3 G^3 \right\rangle}{2}. \end{aligned} \tag{2.60}$$

Then solving this system for our three unknowns A, B and C and substituting the results

back into (2.51) with some simplification will give us:

$$\begin{aligned}
&\mathrm{Tr}\big[\big\langle : G_{\omega\zeta}(0)[D_{\rho_1}(0),[D_{\rho_2}(0),G_{\phi\sigma}(0)]]: \big\rangle\big] = \\
&\qquad = \frac{\langle g_s^3 G^3\rangle}{2d(d^2-4)}\Bigg[-2\, g_{\rho_1\rho_2}(g_{\omega\phi}g_{\zeta\sigma} - g_{\phi\zeta}g_{\omega\sigma}) + \\
&\qquad\quad + \frac{3}{d-1}\,\big[g_{\phi\rho_2}(g_{\omega\rho_1}g_{\zeta\sigma} - g_{\rho_1\zeta}g_{\omega\sigma}) - g_{\sigma\rho_2}(g_{\phi\zeta}g_{\omega\rho_1} - g_{\omega\phi}g_{\zeta\rho_1})\big] + \\
&\qquad\quad + \big[g_{\rho_1\sigma}(g_{\omega\rho_2}g_{\phi\zeta} - g_{\omega\phi}g_{\zeta\rho_2}) - g_{\rho_1\phi}(g_{\omega\rho_2}g_{\zeta\sigma} - g_{\omega\sigma}g_{\rho_2\zeta})\big]\Bigg].
\end{aligned} \tag{2.61}$$

If we now substitute (2.61) into the last term in (2.41), then substitute just this last term into (2.36), we will have the 6D gluon condensate contribution which can be written as follows:

$$\begin{aligned}
\Pi^{\mu\nu}_{(6D)}(q) = &-\frac{i}{8(2\pi)^{3d}}\,\gamma^{\mu}_{ij}\,\gamma^{\rho}_{ks}\,\gamma^{5}_{sm}\,\gamma^{\sigma}_{nr}\,\epsilon^{\nu\rho\omega\zeta}\int d^dx\int d^dz\int d^dp_1\int d^dp_2\int d^dp_3\Bigg(\\
&e^{iz\cdot(p_2-p_3)}e^{ix\cdot(q+p_3-p_1)}S_{ri}(\not{p}_3)S_{jk}(\not{p}_1)S_{mn}(\not{p}_2)\Bigg[-\frac{1}{2}\,z^{\phi}z^{\rho_1}z^{\rho_2}\,\frac{\langle g_s^3G^3\rangle}{2d(d^2-4)}\Bigg[\\
&-2\,g_{\rho_1\rho_2}(g_{\omega\phi}g_{\zeta\sigma} - g_{\phi\zeta}g_{\omega\sigma}) + \frac{3}{d-1}\,\big[g_{\phi\rho_2}(g_{\omega\rho_1}g_{\zeta\sigma} - g_{\rho_1\zeta}g_{\omega\sigma}) - g_{\sigma\rho_2}(g_{\phi\zeta}g_{\omega\rho_1} - g_{\omega\phi}g_{\zeta\rho_1})\big] + \\
&+\big[g_{\rho_1\sigma}(g_{\omega\rho_2}g_{\phi\zeta} - g_{\omega\phi}g_{\zeta\rho_2}) - g_{\rho_1\phi}(g_{\omega\rho_2}g_{\zeta\sigma} - g_{\omega\sigma}g_{\rho_2\zeta})\big]\Bigg]\Bigg]\Bigg).
\end{aligned} \tag{2.62}$$

This expression corresponds to the Feynman diagram represented in Figure 2.4. Once again, we found that the clockwise direction contributes equally to the calculation and so we will need to remember to include an overall factor of two when we write our final contribution from the 6D term.

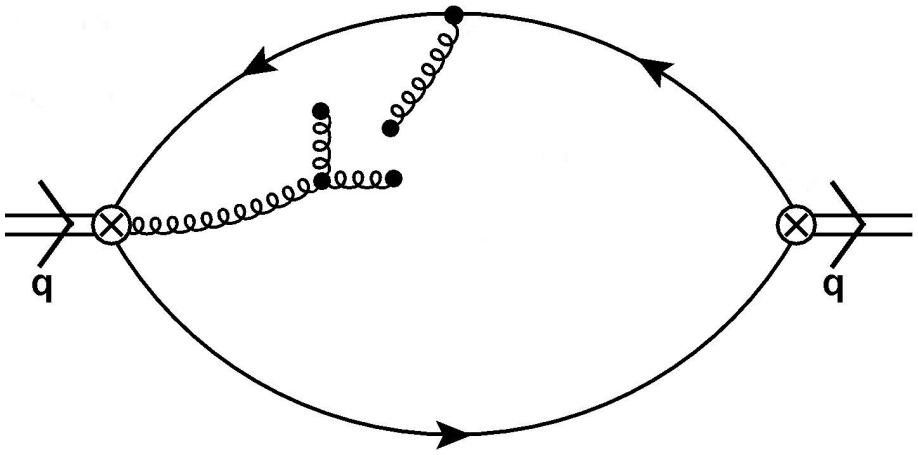

Figure 2.4: Counterclockwise non-perturbative 6D gluon condensate diagram, representing the contribution from equation (2.62).

The simplification of (2.62) is a bit more complicated that the equivalent procedure in the 4D case but the basics steps will be the same. We will again rewrite our factors of z as derivatives of internal momenta applied to our exponentials; in this case we will need three derivatives to account for our three z's. We can again use integration by parts to move these derivatives over to our momentum space propagators. Then by evaluating the appropriate integrals, placing the appropriate terms in a trace and rearranging we can write an equation equivalent to what we wrote in (2.48) but now for the 6D contribution as

$$\begin{aligned}\Pi^{\mu\nu}_{(6D)}(q) = \frac{\langle g_s^3 G^3\rangle}{32d(d^2-4)(2\pi)^d}\epsilon^{\nu\rho\omega\zeta}\int d^d p_3 \Bigg(\\ \mathrm{Tr}\Big[\gamma^\mu\, S(\not{p}_3+\not{q})\,\gamma^\rho\,\gamma^5\,\Big(\frac{\partial}{\partial p_3^\phi}\Big)\Big(\frac{\partial}{\partial p_3^{\rho_1}}\Big)\Big(\frac{\partial}{\partial p_3^{\rho_2}}\Big) S(\not{p}_3)\,\gamma^\sigma\, S(\not{p}_3)\Big]\Big[\\ 2\, g_{\rho_1\rho_2}(g_{\omega\phi}g_{\zeta\sigma} - g_{\phi\zeta}g_{\omega\sigma}) - \frac{3}{d-1}\,\big[g_{\phi\rho_2}(g_{\omega\rho_1}g_{\zeta\sigma} - g_{\rho_1\zeta}g_{\omega\sigma}) - g_{\sigma\rho_2}(g_{\phi\zeta}g_{\omega\rho_1} - g_{\omega\phi}g_{\zeta\rho_1})\big] - \\ -\,\big[g_{\rho_1\sigma}(g_{\omega\rho_2}g_{\phi\zeta} - g_{\omega\phi}g_{\zeta\rho_2}) - g_{\rho_1\phi}(g_{\omega\rho_2}g_{\zeta\sigma} - g_{\omega\sigma}g_{\rho_2\zeta})\big]\Big]\Bigg).\end{aligned} \tag{2.63}$$

The main difference between (2.48) and (2.63) being that, in this case, we have three derivatives acting on our momentum space propagator, not just one. We again have the same

two choices with respect to how we want to evaluate these derivatives. We can use the $\left(\frac{\partial}{\partial p_\phi}\right)S(\not{p}) = S(\not{p})\gamma^\phi S(\not{p})$ identity three times or we can expand the propagator and apply the derivatives directly to the momenta. It turns out however that, in this case, the first technique results in traces of large numbers of gamma matrices. These traces expand into so many terms that the calculation becomes computationally prohibitive and in practice we need to proceed only with the second method.

Like the 4D case, once the derivatives are handled, here the calculation proceeds much like the perturbative case. After simplification we apply the projection operators, again finding the scalar part goes to zero. Continuing with the vector portion we handle the integrals with TARCER and after some simplification we have our 6D contribution.

2.2.7 Exact Leading Order 6D Gluon Condensate Result

Again the result of the above discussed calculation is multiplied by 2 to account for the second contributing contraction direction. Having done so, we find that the exact leading order d-dimensional 6D gluon condensate vector contribution to our correlator is given by:

$$\begin{aligned}\Pi_1^{(6D)}(q) = -2\left\langle g_s^3 G^3\right\rangle \frac{i^{-d}2^{-d-5}(d-3)(d-2)\pi^{-\frac{d}{2}}\left(-m^2\right)^{d/2}\Gamma\left(2-\frac{d}{2}\right)}{(d-1)d(d+2)m^6(z-1)^3 z}\Bigg[\\ \Big(4(d-3)(d+2)z^3+((-3(d-7)d-88)d+148)z^2+\\ +(d-3)((2d-5)d+30)z-12(d-3)\Big)+\\ +\Big((d-3)(z(3(d-6)(d-2)^2z^2-2d((d-9)d+17)z+\\ +(9-2d)d+36z-46)+12)\Big)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};z\right)\Bigg].\end{aligned} \tag{2.64}$$

2.2.8 Epsilon Expanded 6D Result

Just as we did in the perturbative and 4D cases we will now want to ϵ-expand this result. Again following the steps highlighted in the perturbative calculation, we find that the ϵ-expanded 6D vector contribution to our correlator is given by:

$$\Pi^{(6D)}_{1(\epsilon)}(q) = \langle g_s^3 G^3 \rangle \left(- \frac{2\,(8z^3 - 25z^2 + 23z - 6)\,A(z) + 3i\,(4z^3 - 10z^2 + 7z - 2)\,B(z)}{4608\pi^2 m^2 (z-1)^4 A^3(z)} \right). \tag{2.65}$$

It may seem at this point that we are done and, in fact, if we were sufficiently lucky we would be. The full ϵ-expanded vector portion of our correlator will just be the sum of the perturbative, 4D and 6D contributions listed in (2.33), (2.50) and (2.65) respectively. If none of these terms contained problematic divergences, we would now be able to proceed with the use of our dispersion relation and continue on with the sum rules analysis. However, in our case, if we look back at equation (2.33) we noted that it contained a non-polynomial divergence. We will now address this problematic non-polynomial divergence.

Chapter 3

Renormalization

As we mentioned in Chapter 1, our hybrid current (2.4) is not renormalized. This can and, in this case, does result in terms with divergences that we will need to deal with.

3.1 The Need for Renormalization

We start this discussion by closely examining the perturbative, 4D and 6D contributions to our correlator listed in (2.33), (2.50) and (2.65) respectively. We will pay special attention to any terms that diverge as $\epsilon \to 0$. We can see immediately that the 4D and 6D terms have no ϵ dependence but our perturbative term does in the form of terms that go like $\frac{1}{\epsilon}$ or $\frac{1}{\epsilon^2}$, these terms will clearly diverge. However not all divergent terms will be problematic for our sum rules calculation. Remembering the discussion we had about the Borel transform (1.13) in Section 1.2.10, we know that divergences which are polynomial in z (where z is defined in equation (2.26)) will be suppressed by the Borel transform and we only need concern ourselves with non-polynomial divergences. We can see that the part of our perturbative contribution (2.33) that goes like $\frac{1}{\epsilon}$ is unfortunately not polynomial in z. Writing this term out in full after substituting in for the $A(z)$, $B(z)$ and M defined in (2.34) and simplifying gives us:

$$-\frac{1}{\epsilon}\frac{g_s^2 m^4}{18\pi^4}\Bigg(4z^2+12i\pi z+z+6-12z\log\big(-\frac{m^2}{\nu^2}\big)+ \\ +3i(4z+1)\sqrt{\frac{1-z}{z}}\log\Big(\frac{i-\sqrt{\frac{z}{1-z}}}{\sqrt{\frac{z}{1-z}}+i}\Big)\Bigg). \tag{3.1}$$

Written this way, it is clear to see that the last term in (3.1) is giving us our non-polynomial divergence. To write the correlator in a form that is ready for sum rules analysis we will need to deal with this divergence. Said another way, we need to renormalize our perturbative contribution.

3.2 Renormalization

Non-polynomial divergences like (3.1) can be eliminated through operator mixing under renormalization. In doing so we replace a bare current $j^\nu_{(h)}$ (2.4) by the renormalized current $\Big[j^\nu_{(h)}\Big]$. This renormalized current is a linear combination of $j^\nu_{(h)}$ and other operators $\{\mathcal{O}_n\}$ where each $\mathcal{O}_n$ has the same quantum numbers as $j^\nu_{(h)}$ and dimension less than or equal to $j^\nu_{(h)}$ as we described in (1.14).

It is well known that $j^\mu_{(m)}$ is renormalization group invariant and thus $\Big[j^\mu_{(m)}\Big]=j^\mu_{(m)}$. As such, we need only consider the operator mixing of $j^\nu_{(h)}$. As we will demonstrate, it is possible to choose C_1 and C_2 such that

$$\Big[j^\nu_{(h)}\Big]=j^\nu_{(h)}+\frac{C_1}{\epsilon}j^\nu_{(m)}+\frac{C_2}{\epsilon}j^\nu_{(c)} \tag{3.2}$$

leads to a correlator free of non-polynomial divergences at leading order. In (3.2) C_1 and C_2 are $\mathcal{O}(g_s^2)$ coefficients, the current $j^\nu_{(m)}$ is our mesonic current (2.3) and $j^\nu_{(c)}$ which we will refer to as our covariant current is defined by:

$$j^{\nu}_{(c)}(x) = \overline{Q}^{\alpha}_{i}(x) D^{\nu}_{ij} Q^{\alpha}_{j}(x). \tag{3.3}$$

We will now insert (3.2) into a correlation function that will include a mesonic current (2.3) to generate the following expression:

$$\begin{aligned} i\int d^d x\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\, \left[j^{\nu}_{(h)}(0)\right]\, e^{i\int dy \mathcal{L}_{int}(y)}]|\Omega\rangle = \\ = i\int d^d x\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\, j^{\nu}_{(h)}(0)\, e^{i\int dy \mathcal{L}_{int}(y)}]|\Omega\rangle + \\ + \frac{C_1}{\epsilon}\, i\int d^d x\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\, j^{\nu}_{(m)}(0)\, e^{i\int dy \mathcal{L}_{int}(y)}]|\Omega\rangle + \\ + \frac{C_2}{\epsilon}\, i\int d^d x\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\, j^{\nu}_{(c)}(0)\, e^{i\int dy \mathcal{L}_{int}(y)}]|\Omega\rangle. \end{aligned} \tag{3.4}$$

The first term on the RHS of (3.4) was the subject of most of Section 2.1. We will now compute the other two terms. Lastly we will tune the constants C_1 and C_2 to eliminate all non-polynomial divergences.

3.2.1 Meson - Meson Calculation

We will now be interested in calculating the perturbative ϵ-expanded contribution resulting from the following expression:

$$\Pi^{\mu\nu}_{(mm)}(q) = i\int d^d x\; e^{iq\cdot x}\langle\Omega|T[\, j^{\mu}_{(m)}(x)\, j^{\nu}_{(m)}(0)\, e^{i\int dy \mathcal{L}_{int}(y)}]|\Omega\rangle \tag{3.5}$$

This expression corresponds to the diagram shown in (3.1). Note that at leading order $e^{i\int dy \mathcal{L}_{int}(y)} \to 1$ in this case.

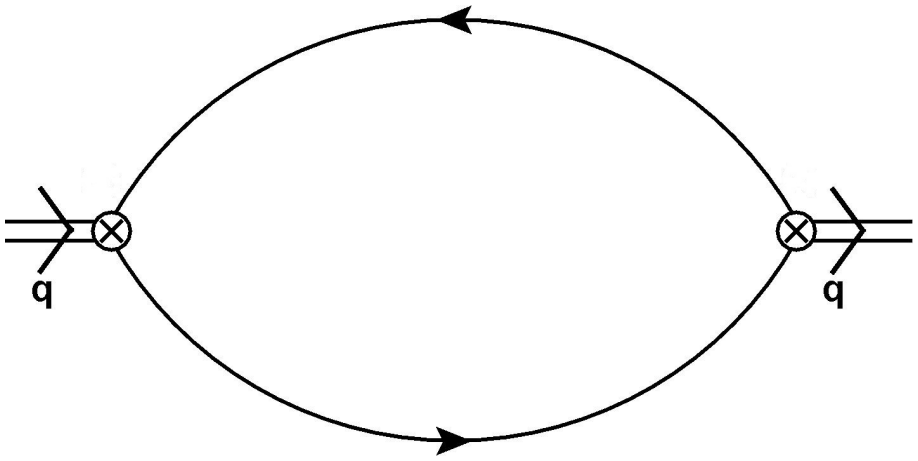

Figure 3.1: Counterclockwise perturbative meson - meson diagram.

We now proceed by substituting two copies of (2.3) into (3.5). After following all the steps we highlighted in Section 2.1 to simplify this expression, we find that the exact leading order d-dimensional contribution is given by:

$$\Pi^{(pert)}_{(mm)}(q) = \frac{48(\sqrt{\pi})^d m^{d-2}}{(2\pi)^d(d-1)}\Gamma\Big(2-\frac{d}{2}\Big)\bigg(-1+\Big(1+(d-2)z\Big)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};z\right)\bigg). \quad (3.6)$$

Expanding in ϵ gives us:

$$\begin{aligned}\Pi^{(pert)}_{(mm)(\epsilon)}(q) = -\frac{m^2 z}{\pi^2\epsilon} + \frac{m^2}{6\pi^2 A(z)}\Bigg(&3i(2z+1)B(z)+\\ &+2A(z)\Big(-3z\log(M)+(5+3i\pi)z+3\Big)\Bigg).\end{aligned} \quad (3.7)$$

3.2.2 Meson - Covariant Current Calculation

We now want to calculate the perturbative ϵ-expanded contribution resulting from

$$\Pi^{\mu\nu}_{(mc)}(q) = i\int d^dx\; e^{iq\cdot x}\langle\Omega|T[\,j^{\mu}_{(m)}(x)\; j^{\nu}_{(c)}(0)\; e^{i\int dy\mathcal{L}_{int}(y)}]|\Omega\rangle. \tag{3.8}$$

We proceed by substituting (2.3) and (3.3) into (3.5). Note that the diagram resulting from this expression would be identical to Figure 3.1 so we will not redraw it here. Again at leading order $e^{i\int dy\mathcal{L}_{int}(y)} \to 1$ here. After simplification we find that the exact leading order result is given by:

$$\Pi^{(pert)}_{(mc)}(q) = \frac{24i(\sqrt{\pi})^d m^{d-1}}{(2\pi)^d(d-1)}\Gamma\Big(1-\frac{d}{2}\Big)\left(1-(d-2)(z-1)\,{}_2F_1\left(1,2-\frac{d}{2};\frac{3}{2};z\right)\right). \tag{3.9}$$

Then, ϵ-expanding this result gives us:

$$\begin{aligned}\Pi^{(pert)}_{(mc)(\epsilon)}(q) = \frac{im^3(3-2z)}{4\pi^2\epsilon} - \frac{m^3}{12\pi^2 A(z)}\Bigg(&6(z-1)B(z) - iA(z)\Big(3\gamma_E - \\ &- 6(z-1)\log(M) + 6i\pi(z-1) + 16z - 19\Big)\Bigg).\end{aligned} \tag{3.10}$$

3.2.3 Continuing Renormalization

Using the results (3.7), (3.10) and (2.33), we can identify the $\frac{1}{\epsilon}$ term from (3.4) which we find is given by:

$$\frac{m^2}{36\pi^4\epsilon A(z)}\Bigg(-3iB(z)\Big(g_s^2m^2(4z+1)-6\pi^2\Big(im(z-1)C_2+2zC_1+C_1\Big)\Big)+ \\ +A(z)\Bigg(-g_s^2m^2\Big(4z^2+12i\pi z+z+6\Big)+12\pi^2(3+(5+3i\pi)z)C_1+ \\ +3\pi^2m(-(6\pi z)+16iz+6\pi+3i\gamma_E-19i)C_2+ \\ +6\log(M)\Big(2g_s^2m^2z+3\pi^2(-2zC_1-im(z-1)C_2)\Big)\Bigg)\Bigg) \tag{3.11}$$

If we now ignore polynomials in z (which will ultimately be suppressed by the Borel transform), (3.11) reduces to:

$$\frac{i(z-1)}{12\pi^4z}\Bigg(g_s^2m^2(4z+1)-6\pi^2\Big(im(z-1)C_2+2zC_1+C_1\Big)\Bigg). \tag{3.12}$$

Expression (3.12) must go to zero for all z to eliminate all non-polynomial divergences. Setting (3.12) to zero, expanding and solving for C_1 and C_2 gives us:

$$\begin{aligned} C_1 &= \frac{5g_s^2m^2}{18\pi^2} \\ C_2 &= -\frac{ig_s^2m}{9\pi^2}. \end{aligned} \tag{3.13}$$

Writing out (3.4) in full, using (3.7), (3.10), (2.33) and (3.13), then simplifying and suppressing the square brackets that indicate a renormalized quantity, we find that:

$$
\begin{aligned}
\Pi^{(pert)}_{1(\epsilon)}(q) =& \frac{g_s^2 m^4}{432\pi^4}\Bigg(\frac{36-72z}{\epsilon^2}+ \\
&+\frac{4}{\epsilon}\Big(6\log(M)+3(21-4z)z-6i\pi+3\gamma_E-7\Big)+ \\
&+\frac{1}{zA(z)}\Bigg(9\big(8z^2-4z+1\big)A(z)B^2(z)+ \\
&\quad+12izB(z)\Big(-6(4z+1)\log(M)+8z^2+6z+6i\pi(4z+1)-1\Big)+ \\
&\quad+2z\Big(12i\pi\left(4z^2+z+6\right)+(76z+61)z-66\pi^2z+6\Big)A(z)- \\
&\quad-24zA(z)\log(M)\Big(-6z\log(M)+4z^2+12i\pi z+z+6\Big)+ \\
&\quad+18iz(4z+1)\Big(\big(\log(1+iA(z))+\log(\frac{1}{4}(1-iA(z)))\big)\times \\
&\qquad\times\big(\log(1+iA(z))-\log(1-iA(z))\big)\Big)+ \\
&\quad+36iz(4z+1)\Big(\mathrm{Li}_2\big(\frac{1}{2}(iA(z)+1)\big)-\mathrm{Li}_2\big(\frac{1}{2}(1-iA(z))\big)\Big)\Bigg)\Bigg).
\end{aligned}
\tag{3.14}
$$

Examining (3.14) more closely, we can see that we have successfully eliminated all non-polynomial divergences.

Chapter 4

Conclusion

Now that the calculation of our cross-correlator is complete and all the results have been renormalized we can summarize our results.

4.1 Summary of Results

Where possible, we will write our results in terms of the strong coupling constant α_s where

$$\alpha_s = \frac{g_s^2}{4\pi} \tag{4.1}$$

the choice to use α_s in writing these results is customary for this kind of calculation. To clean up the notation, we will continue to use $A(z)$, $B(z)$ and M which are given in (2.34). We rewrite their definitions here for the reader's convenience:

$$\begin{aligned} A(z) &= \sqrt{\frac{z}{1-z}} \\ B(z) &= \log\left(\frac{i - A(z)i}{A(z) + i}\right) \\ M &= -\frac{m^2}{\nu^2}. \end{aligned} \tag{4.2}$$

where ν is the mass parameter we introduced in dim-reg. We also rewrite our 4D and 6D gluon condensates which are defined in (2.46) and (2.59) in terms of α_s (where possible)

$$\begin{aligned}\left\langle \alpha_s G^2 \right\rangle &= \alpha_s \big\langle : G^a_{\omega\phi} G^a_{\omega\phi} : \big\rangle \\ \left\langle g_s^3 G^3 \right\rangle &= g_s^3 f_{abc} \big\langle : G^a_{\omega\zeta}\, G^b_{\zeta\rho_1} G^c_{\rho_1\omega} : \big\rangle\end{aligned} \tag{4.3}$$

note that the factor of 4π in the 4D case will be absorbed into our final result for the 4D contribution. We continue to define z as:

$$z = \frac{q \cdot q}{4m^2} \tag{4.4}$$

and $\mathrm{Li}_s(z)$ functions continue to represent polylogarithms which are given by:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{4.5}$$

Finally remembering that γ_E is Euler's constant given by:

$$\gamma_E = \lim_{n\to\infty} \left(-\ln n + \sum_{n}^{k=1} \frac{1}{k} \right) = 0.57722\ldots \tag{4.6}$$

we can write our final results as follows:

Perturbative Result

$$\begin{aligned}\Pi^{(pert)}_{1(\epsilon)}(q) =&\frac{\alpha_s m^4}{108\pi^3}\Bigg(\frac{36-72z}{\epsilon^2}+\\
&+\frac{4}{\epsilon}\Big(6\log(M)+3(21-4z)z-6i\pi+3\gamma_E-7\Big)+\\
&+\frac{1}{zA(z)}\Bigg(9\Big(8z^2-4z+1\Big)A(z)B^2(z)+\\
&\quad+12izB(z)\Big(-6(4z+1)\log(M)+8z^2+6z+6i\pi(4z+1)-1\Big)+\\
&\quad+2z\Big(12i\pi\left(4z^2+z+6\right)+(76z+61)z-66\pi^2z+6\Big)A(z)-\\
&\quad-24zA(z)\log(M)\Big(-6z\log(M)+4z^2+12i\pi z+z+6\Big)+\\
&\quad+18iz(4z+1)\Big(\big(\log(1+iA(z))+\log(\frac{1}{4}(1-iA(z)))\big)\times\\
&\qquad\times\big(\log(1+iA(z))-\log(1-iA(z))\big)\Big)+\\
&\quad+36iz(4z+1)\Big(\mathrm{Li}_2\big(\frac{1}{2}(iA(z)+1)\big)-\mathrm{Li}_2\big(\frac{1}{2}(1-iA(z))\big)\Big)\Bigg)\Bigg)\end{aligned} \tag{4.7}$$

4D Result

$$\Pi^{(4D)}_{1(\epsilon)}(q)=\left\langle\alpha_s G^2\right\rangle\frac{2z+iA(z)B(z)}{12\pi z} \tag{4.8}$$

6D Result

$$\Pi^{(6D)}_{1(\epsilon)}(q)=\left\langle g_s^3G^3\right\rangle\left(\frac{2\left(-8z^3+25z^2-23z+6\right)A(z)-3i\left(4z^3-10z^2+7z-2\right)B(z)}{4608\pi^2m^2(z-1)^4A^3(z)}\right) \tag{4.9}$$

4.2 Closing Thoughts

4.2.1 Error Checking

With a project of this size there are a number of ways in which errors could be introduced into the calculation. To eliminate these errors this calculation was done by two different people using different software where possible; the results were compared and found to agree. In addition each individual's code has been tested against known results and where ever possible new results are tested by hand. Finally when ever possible consistency checks are made against expected findings, such as the verification that $\Pi_0(q) \to 0$ mentioned in Section (2.1.5).

4.2.2 Future Calculations

Now that we have our cross-correlator written in this form, we are ready to move on to the next step in our exploration of vector charmonium meson-hybrid mixing. As we mentioned in Chapter 1, this would entail inserting our results into the dispersion relation (1.10). Then we would proceed with the QCD sum rules analysis which would allow us to get at the particle's physical mass.

Considering the predicted mass from the charmonium 1^{--} hybrid current [19] at 3.36 ± 0.15 GeV and that the resonance that has conventionally been assigned to the pure ψ' has a mass of around 3.686 GeV, we feel there is good reason to expect the charmonium meson-hybrid mix to come out somewhere close to this mass. If this turns out to be the case this would indicate that perhaps these hybrids exist primarily as quantum mechanical superpositions of pure mesonic and hybrid states. If, however, we find a mass that falls well outside of this range it would raise interesting questions as well. Doing the sum rules portion of the calculation is the next item on the agenda so that we can try to answer some of these questions.

4.2.3 Other Questions and Ideas

There are a few other interesting checks and comparisons that could be made using the results of our correlator calculation. Even though the renormalization scheme we employed in our calculation is widely accepted and used, it would be interesting to do the work required to fully renormalize the hybrid current (2.4). We could then run the correlator calculation again using the renormalized current to make sure the results agree.

In doing this correlator calculation, we have adopted one convention for writing our γ^5 in d-dimensions consistent with [23]. There are other conventions that could be used but at two-loops it is widely believed [13] that that all γ^5 conventions will agree. This would nonetheless be interesting to verify.

Another interesting possibility would be the addition of higher loop corrections to the calculation. The calculations needed to determine these contributions would require the solving of integrals with no known analytical solution but perhaps numerical methods could be used to solve this problem. Adding in these contributions would correspond to keeping an additional term in (2.2).

Finally, it might be interesting to add in 8D condensate terms; such terms would be heavily suppressed by factors of one over the charm quark mass and one over our external momentum. It is probably safe to assume such a term would contribute next to nothing but it would be interesting to check. Adding in these contributions would correspond to keeping an additional term in (2.41).

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Appendix A

Identities and Definitions

This appendix will serve as a collection of several of the identities and definitions commonly used in calculations of this kind.

A.1 Colour Algebra

With regard to the use of δ functions we use a convention where a delta function appearing with Latin indices is understood to exist in gluon colour space such that

$$\delta^{ab}\delta_{ab} = \delta^a_a = 8 \tag{A.1}$$

and a delta function appearing with Greek indices is understood to exist in quark colour space such that

$$\delta_{\alpha\beta}\delta^{\alpha\beta} = \delta^\alpha_\alpha = 3 \tag{A.2}$$

finally a delta function appearing with functional arguments (such as $\delta(p-q)$) is understood to be a d dimensional Dirac delta function.

Lambdas (λ) will be understood to be Gell-Mann matrices and they relate to the generators of $SU(3)$ (which we represent with t) as follows:

$$\lambda^a = 2t^a. \tag{A.3}$$

The traces of these Gell-Mann matrices obey the following identities, additional identities can be found in [11]

$$Tr[\lambda^a] = 0 \tag{A.4}$$

$$\lambda^a_{\alpha\beta}\lambda^b_{\beta\alpha} = Tr[\lambda^a\lambda^b] = 2\delta^{ab} \tag{A.5}$$

$$\lambda^a_{\alpha\beta}\lambda^b_{\beta\delta}\lambda^c_{\delta\alpha} = Tr[\lambda^a\lambda^b\lambda^c] = 2(d^{abc} + if^{abc}) \tag{A.6}$$

where the d^{abc} are totally symmetric and the f^{abc} are totally antisymmetric structure constants.

A.2 Dirac Algebra

We use the mostly minus sign convention for our Minkowski metric tensors (g) and in d-dimensions they obey the following identities:

$$g^{\mu\nu}g_{\mu\nu} = g^\mu_\mu = d \tag{A.7}$$

and

$$g^{\mu\nu}p_\mu q_\nu = p^\nu q_\nu = p \cdot q \tag{A.8}$$

where p and q are some four-momenta. We define γ^5 in d-dimensions by the following convention:

$$\gamma^5 = -\frac{i}{24}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho \tag{A.9}$$

which is consistent with [23]. With regard to the traces of gamma matrices we use the following identities. Additional identities involving traces without γ^5's in them can be found in [10] and can be handled by our code. For the sake of brevity we omit identities with larger traces.

$$Tr[\gamma^5] = 0 \tag{A.10}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^5] = 0 \tag{A.11}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma} \tag{A.12}$$

$$
\begin{aligned}
Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\eta\gamma^\tau\gamma^5] = -4i\Big(&\epsilon^{\mu\nu\rho\sigma}g^{\eta\tau} - \epsilon^{\nu\rho\sigma\tau}g^{\mu\eta} + \epsilon^{\rho\sigma\eta\tau}g^{\mu\nu} - \epsilon^{\nu\sigma\eta\tau}g^{\mu\rho} + \\
&\epsilon^{\nu\rho\eta\tau}g^{\mu\sigma} + \epsilon^{\nu\rho\sigma\eta}g^{\mu\tau} + \epsilon^{\mu\rho\sigma\tau}g^{\nu\eta} + \epsilon^{\mu\sigma\eta\tau}g^{\nu\rho} - \\
&\epsilon^{\mu\rho\eta\tau}g^{\nu\sigma} - \epsilon^{\mu\rho\sigma\eta}g^{\nu\tau} - \epsilon^{\mu\nu\sigma\tau}g^{\rho\eta} + \epsilon^{\mu\nu\eta\tau}g^{\rho\sigma} + \\
&\epsilon^{\mu\nu\sigma\eta}g^{\rho\tau} + \epsilon^{\mu\nu\rho\tau}g^{\sigma\eta} - \epsilon^{\mu\nu\rho\eta}g^{\sigma\tau}\Big)
\end{aligned}
\tag{A.13}
$$

The most general contraction of two Levi-Civita symbols can be written as follows. Similar (but more compact) identities where the Levi-Civita symbols share some indices can be found in [10] among other places

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} = &- g^{\alpha\sigma}g^{\beta\rho}g^{\gamma\nu}g^{\delta\mu} + g^{\alpha\rho}g^{\beta\sigma}g^{\gamma\nu}g^{\delta\mu} + g^{\alpha\sigma}g^{\beta\nu}g^{\gamma\rho}g^{\delta\mu} - g^{\alpha\nu}g^{\beta\sigma}g^{\gamma\rho}g^{\delta\mu} \\
&- g^{\alpha\rho}g^{\beta\nu}g^{\gamma\sigma}g^{\delta\mu} + g^{\alpha\nu}g^{\beta\rho}g^{\gamma\sigma}g^{\delta\mu} + g^{\alpha\sigma}g^{\beta\rho}g^{\gamma\mu}g^{\delta\nu} - g^{\alpha\rho}g^{\beta\sigma}g^{\gamma\mu}g^{\delta\nu} \\
&- g^{\alpha\sigma}g^{\beta\mu}g^{\gamma\rho}g^{\delta\nu} + g^{\alpha\mu}g^{\beta\sigma}g^{\gamma\rho}g^{\delta\nu} + g^{\alpha\rho}g^{\beta\mu}g^{\gamma\sigma}g^{\delta\nu} - g^{\alpha\mu}g^{\beta\rho}g^{\gamma\sigma}g^{\delta\nu} \\
&- g^{\alpha\sigma}g^{\beta\nu}g^{\gamma\mu}g^{\delta\rho} + g^{\alpha\nu}g^{\beta\sigma}g^{\gamma\mu}g^{\delta\rho} + g^{\alpha\sigma}g^{\beta\mu}g^{\gamma\nu}g^{\delta\rho} - g^{\alpha\mu}g^{\beta\sigma}g^{\gamma\nu}g^{\delta\rho} \\
&- g^{\alpha\nu}g^{\beta\mu}g^{\gamma\sigma}g^{\delta\rho} + g^{\alpha\mu}g^{\beta\nu}g^{\gamma\sigma}g^{\delta\rho} + g^{\alpha\rho}g^{\beta\nu}g^{\gamma\mu}g^{\delta\sigma} - g^{\alpha\nu}g^{\beta\rho}g^{\gamma\mu}g^{\delta\sigma} \\
&- g^{\alpha\rho}g^{\beta\mu}g^{\gamma\nu}g^{\delta\sigma} + g^{\alpha\mu}g^{\beta\rho}g^{\gamma\nu}g^{\delta\sigma} + g^{\alpha\nu}g^{\beta\mu}g^{\gamma\rho}g^{\delta\sigma} - g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}.
\end{aligned}
\tag{A.14}
$$

Appendix B

Mathematica Code

This appendix contains the main body of the Mathematica code we have used to do our correlator calculation. This section of code when executed in the correct order will take in correlators in the form of (2.5) and return ϵ-expanded results much like those seen in (2.33) or (2.50). It is important to note that this process is not fully automated; the user makes a number of choices in what functions to apply at what times. This allows for a number of double checks and the production of intermediate results for verification purposes.

This section of code makes use of the TARCER Mathematica package that was discussed in Section (2.1.6) and the HypExp package mentioned in Section (2.1.8). Note that there are additional peripheral code blocks that were used to generate results used in this code that are not included here. Additionally we have omitted certain sections of code that are too large to include here, such as the libraries of dim-reg integral identities or the library of γ^5 trace identities. Finally, it should be mentioned that none of the code used to implement our renormalization scheme is included here either. These code blocks were omitted in an attempt to minimize the size of this document.

B.1 Main Mathematica Code

CORRELATION FUNCTION CALCULATOR

Alex Palameta, University of Saskatchewan

Clear and Imports

```
ClearAll["Global`*"]
Import["C:\\ACFC\\tarcerWindows64bit25.mx"]
Get[ToFileName[{"C:\\ACFC\\HypExp-win-2.0"}, "init.m"]] ;
<< HypExp`;
Get[ToFileName[{ "C:\\ACFC\\HypExp-win-2.0"}, "install_lib3.m"]]
math < install_lib3.m;
Get[ToFileName[{ "C:\\ACFC\\HypExp-win-2.0"}, "install_lib2.m"]]
math < install_lib2.m;
```

Attributes and Abbreviations

Attributes

```
$RecursionLimit = 10 000;
Clear[metric, delta3, delta8, dim, DIM]
SetAttributes[metric, Orderless]
SetAttributes[delta3, Orderless]
SetAttributes[delta8, Orderless]
SetAttributes[Dot, Orderless]
metric[μ_, μ_] := DIM;
delta3[α_, α_] = 3;
delta8[a_, a_] = 8;
dim = 4 + 2 ϵ;
```

Format

Abbreviations

```
Clear[NCM, Γ, HPFQ, TForm]
NCM = NonCommutativeMultiply;
Γ = Gamma;
HPFQ = HypergeometricPFQ;
TForm = TraditionalForm;
```

Simplification Algorithms

splitAndFactorOutOfNCM[x_]

```
Clear[splitAndFactorOutOfNCM, splitNCM]
splitAndFactorOutOfNCM[x_] := Module[{in},
  in = x //. NCM[a___, b_, c___] :> b NCM[a, c] /; FreeQ[b, qqq, {0, Infinity}] &&
       FreeQ[b, qqqB, {0, Infinity}] && FreeQ[b, GGG, {0, Infinity}] &&
       FreeQ[b, AAA, {0, Infinity}] && FreeQ[b, PD, {0, Infinity}];
  in = splitNCM[in];
  in
 ]
splitNCM[x_] := Module[{in, qL, gL, out},
  in = x //. b__ NCM[a___] :> List@a;
  gL = DeleteCases[in, _qqq];
  gL = DeleteCases[gL, _PD];
  gL = NCM @@ DeleteCases[gL, _qqqB];
  qL = DeleteCases[in, _GGG];
  qL = NCM @@ DeleteCases[qL, _AAA];
  out = x /. a___ NCM[b___] :> a qL gL;
  out
 ]
```

simplifyExp[x_]

```
Clear[chopBigDeltaMinus, simplifyExp]
chopBigDeltaMinus[y_] := Module[{in, s = SPT},
  in = y //. DDelta4[a_ + b_ + c_ + d_] :> INTEG[s] DDelta4[s + a + b] DDelta4[s - c - d];
  in = in //. INTEG[-a_] :> INTEG[a];
  in
 ]
simplifyExp[y_] := Module[{in},
  in = y //. EXP[-i, k_, x_ - z_] :> EXP[i, k, z - x];
  in = in //. EXP[i, k_, x_ - z_] :> EXP[i, k, x] EXP[i, -k, z];
  in = in //. EXP[i, k_, o] :> 1;
  in = in //. EXP[-i, k_, o] :> 1;
  in = in //. EXP[i, k_, 0] :> 1;
  in = in //. EXP[-i, k_, 0] :> 1;
  in = in //. EXP[i, a_, x_] EXP[i, b__, x_] :> EXP[i, a + b, x];
  in = in //. INTEG[x_] EXP[i, b__, x_] :> (2 π)^DIM DDelta4[b];
  in = chopBigDeltaMinus[in]
 ]
```

setUpTrace[x_]

```
Clear[setUpTrace, deltaContract, orderStuff, expandSS, cleanUpNCM]
setUpTrace[x_] := Module[{in},
  in = x;
  in = Expand[Distribute[in //. mom[(p_ + q_)[μ_]] :> mom[p[μ]] + mom[q[μ]]]] //.
    mom[(-p_)[μ_]] :> -mom[p[μ]];
  in = deltaContract[in];
  in = orderStuff[in];
```

```
  in = expandSS[in];
  in = fixUpNCM[in];
  in
 ]
deltaContract[y_] := Module[{in},
  in = y //. λλλ[a_, α_, β_] :→ 2 ttt[a, α, β];
  in = in //. delta3[α_, β_] ttt[a_, α_, γ_] :→ ttt[a, β, γ];
  in = in //. delta3[α_, β_] ttt[a_, γ_, α_] :→ ttt[a, γ, β];
  in = in //. delta8[a_, b_] ttt[a_, α_, β_] :→ ttt[b, α, β];
  in = in //. ttt[a_, ν_, μ_] ttt[b_, μ_, ν_] :→ 1/2 delta8[a, b];
  (* P&S (A.34) with c(r)=1/2 *)
  in = in //. ttt[a_, μ_, ν_] ttt[a_, ν_, ρ_] :→ 4/3 delta3[μ, ρ];
  (* P&T (A.19) with N=3 *)
  in = in //. ttt[a_, ν_, μ_] ttt[a_, ρ_, ν_] :→ 4/3 delta3[μ, ρ];
  (* P&T (A.19) with N=3 (same just fliped) *)
  in
 ]
orderStuff[y_] := Module[{in},
  in = y //. γγγ[μ_, i_, j_] :→ ord[i, j, dm[γ[μ]]] //.
      γγγ5[i_, j_] :→ ord[i, j, dm[γ[5]]] //.
     SSS[p_, i_, j_] :→ ord[i, j, dm[SS[p]]] //. III[i_, j_] :→ ord[i, j, 1];
  in = Expand[in];
  in = in //. ord[i_, j_, x_] ord[j_, k_, z_] :→ ord[i, k, x ** z] //.
      ord[i_, j_, 1 ** x_] :→ ord[i, j, x] //. ord[i_, j_, x_ ** 1] :→ ord[i, j, x] //.
    ord[i_, j_, x_ ** 1 ** z_] :→ ord[i, j, x ** z];
  in = in //. ord[i_, i_, x_] :→ x;
  in
 ]
expandSS[y_] := Module[{in},
  in = y //.
    dm[SS[SLASH[p__]]] :→ Module[{μ = Unique["μ"]}, (dm[γ[μ]] mom[p[μ]] + mass)/(p.p - mass^2)];
  in
 ]
cleanUpNCM[y_] := Module[{in},
  in = Expand[Distribute[y //. mom[(p_ + q_)[μ_]] :→ mom[p[μ]] + mom[q[μ]]]] //.
    mom[(-p_)[μ_]] :→ -mom[p[μ]];
  in = Expand[Distribute[in]] //. NCM[a___, b_, c___] :→
     b NCM[a, c] /; FreeQ[b, dm];
  in = Expand[Distribute[in //. NCM[a___, (b_ z_), c___] :→
       b NCM[a, z, c] /; FreeQ[b, dm]]];
  in = Expand[Distribute[in //. NCM[z___, (a__ + b__), s___] :→
       NCM[z, a, s] + NCM[z, b, s]]];
  in = Expand[Distribute[in //. NCM[z___, (r__ (a__ + b__)), s___] :→
       NCM[z, r, a, s] + NCM[z, r, b, s]]];
  in = in //. metric[μ_, ν_]^2 :→ DIM //. delta3[α_, β_]^2 :→ 3 //. delta8[a_, b_]^2 :→ 8;
```

```
  in
 ]
fixUpNCM[y_] := FixedPoint[cleanUpNCM, y]
```

■ cleanUp LCandMetric[x_]

```
Clear[leviCivita, cleanUpRules, cleanUpLCandMetric]
leviCivita[x_] := Module[{in, out},
  in = x //. ϵϵϵ[a_, b_, c_, d_] ϵϵϵ[i_, j_, k_, l_] :>
      -(metric[a, l] metric[b, k] metric[c, j] metric[d, i] -
         metric[a, k] metric[b, l] metric[c, j] metric[d, i] -
         metric[a, l] metric[b, j] metric[c, k] metric[d, i] +
         metric[a, j] metric[b, l] metric[c, k] metric[d, i] +
         metric[a, k] metric[b, j] metric[c, l] metric[d, i] -
         metric[a, j] metric[b, k] metric[c, l] metric[d, i] -
         metric[a, l] metric[b, k] metric[c, i] metric[d, j] +
         metric[a, k] metric[b, l] metric[c, i] metric[d, j] +
         metric[a, l] metric[b, i] metric[c, k] metric[d, j] -
         metric[a, i] metric[b, l] metric[c, k] metric[d, j] -
         metric[a, k] metric[b, i] metric[c, l] metric[d, j] +
         metric[a, i] metric[b, k] metric[c, l] metric[d, j] +
         metric[a, l] metric[b, j] metric[c, i] metric[d, k] -
         metric[a, j] metric[b, l] metric[c, i] metric[d, k] -
         metric[a, l] metric[b, i] metric[c, j] metric[d, k] +
         metric[a, i] metric[b, l] metric[c, j] metric[d, k] +
         metric[a, j] metric[b, i] metric[c, l] metric[d, k] -
         metric[a, i] metric[b, j] metric[c, l] metric[d, k] -
         metric[a, k] metric[b, j] metric[c, i] metric[d, l] +
         metric[a, j] metric[b, k] metric[c, i] metric[d, l] +
         metric[a, k] metric[b, i] metric[c, j] metric[d, l] -
         metric[a, i] metric[b, k] metric[c, j] metric[d, l] -
         metric[a, j] metric[b, i] metric[c, k] metric[d, l] +
         metric[a, i] metric[b, j] metric[c, k] metric[d, l]);
  out = Expand[Distribute[in]];
  out
 ]
cleanUpRules[x_] := Module[{in},
  in = x //. metric[μ_, ν_]^2 :> DIM;
  in = in //. metric[μ_, ν_] metric[ν_, ρ_] :> metric[μ, ρ];
  in = in //. metric[μ_, ν_] mom[p_[μ_]] :> mom[p[ν]];
  in = in //. mom[p_[μ_]]^2 :> Dot[p, p];
  in = in //. mom[p_[μ_]] mom[q_[μ_]] :> Dot[p, q];
  in
 ]
cleanUpLCandMetric[x_] := Module[{in},
  in = leviCivita[x];
  in = FixedPoint[cleanUpRules, in];
  in
 ]
```

Contraction Algorithms

Q Contractions (Fermi)

■ contractQql[x_]

```
Clear[contractQql]

contractQql[y_] := Module[{in},
  in = y //. a___ ** qqq[x_, α_, i_] ** qqqB[v_, β_, j_] ** c___ :>
     Module[ {p = Unique["p"]}, i delta3[α, β] 1/(2 π)^DIM INTEG[p]
        EXP[-i, p, x - v] SSS[SLASH[p], i, j] a ** c /; UnsameQ[x, v]];
  in = in //. a___ ** qqqB[v_, β_, j_] ** qqq[x_, α_, i_] ** c___ :>
     Module[ {p = Unique["p"]}, -i delta3[α, β] 1/(2 π)^DIM INTEG[p]
        EXP[-i, p, x - v] SSS[SLASH[p], i, j] a ** c /; UnsameQ[x, v]];
  in = in /. NCM[] :> 1;
  in = in //. NCM[a___, 1, b___] :> NCM[a, b]
 ]
```

■ contractQqbl[x_]

```
Clear[contractQqbl]

contractQqbl[y_] := Module[{in},
  in = y //.
    a___ ** qqqB[v_, β_, j_] ** qqq[b__] ** qqqB[c__] ** qqq[x_, α_, i_] ** d___ :>
     Module[ {p = Unique["p"]}, -i delta3[α, β] 1/(2 π)^DIM INTEG[p] EXP[-i, p, x - v]
        SSS[SLASH[p], i, j] a ** qqq[b] ** qqqB[c] ** d /; UnsameQ[x, v]];
  in = in //. a___ ** qqqB[v_, β_, j_] ** qqqB[c__] ** qqq[x_, α_, i_] ** d___ :>
     Module[ {p = Unique["p"]}, i delta3[α, β] 1/(2 π)^DIM INTEG[p] EXP[-i, p, x - v]
        SSS[SLASH[p], i, j] a ** qqqB[c] ** d /; UnsameQ[x, v]];
  in = in //. a___ ** qqq[x_, α_, i_] ** qqqB[v_, β_, j_] ** c___ :>
     Module[ {p = Unique["p"]}, i delta3[α, β] 1/(2 π)^DIM INTEG[p]
        EXP[-i, p, x - v] SSS[SLASH[p], i, j] a ** c /; UnsameQ[x, v]];
  in = in /. NCM[] :> 1;
  in = in //. NCM[a___, 1, b___] :> NCM[a, b]
 ]
```

■ contractdQql[x_]

```
Clear[contractdQql]

contractdQql[y_] := Module[{in},
  in = y //. a___ ** PD[ν_] ** qqq[x_, α_, i_] ** qqqB[ν_, β_, j_] ** c___ :>
     Module[ {p = Unique["p"]}, mom[p[ν]] delta3[α, β] 1/(2 π)^DIM INTEG[p]
        EXP[-i, p, x - v] SSS[SLASH[p], i, j] a ** c /; UnsameQ[x, v]];
  in = in //. a___ ** qqqB[ν_, β_, j_] ** qqq[x_, α_, i_] ** c___ :>
     Module[ {p = Unique["p"]}, -i delta3[α, β] 1/(2 π)^DIM INTEG[p]
        EXP[-i, p, x - v] SSS[SLASH[p], i, j] a ** c /; UnsameQ[x, v]];
  in = in /. NCM[] :> 1;
  in = in //. NCM[a___, 1, b___] :> NCM[a, b]
 ]
```

G/A Contractions (Bose)

■ contractA[x_]

```
Clear[sendGsToAs, contractA]

sendGsToAs[y_] := Module[{in},
  in = y /. GGG[x_, a_, ω_, f_] :>
     Module[{b = Unique["b"], c = Unique["c"]}, DDD[ω, AAA[x, a, f]] -
       DDD[f, AAA[x, a, ω]] + gs fff[a, b, c] AAA[x, b, ω] ** AAA[x, c, f]];
  in = in //. NCM[(a_ + b_ + c_), (d_ + e_ + f_)] :>
     a ** d + a ** e + a ** f + b ** d + b ** e + b ** f + c ** d + c ** e + c ** f;
  in = in //. NCM[i_ + j_, k_] :> NCM[i, k] + NCM[j, k];
  in = in //. NCM[i___ NCM[j___], k___] :> i * NCM[j, k];
  in = in //. NCM[k___, i___ NCM[j___]] :> i * NCM[k, j];
  in = Expand[Distribute[in //. NCM[a___, (b_ z_), c___] :>
       b NCM[a, z, c] /; FreeQ[b, AAA[___], {0, Infinity}]]];
  in = Expand[in]
 ]

contractA[y_] := Module[{in},
  in = sendGsToAs[y];
  in = in //. NCM[AAA[___], AAA[___], AAA[___], AAA[___]] :> 0;
  in = in //. NCM[AAA[___], AAA[___], AAA[___]] :> 0;
  in = in //. NCM[DDD[___], AAA[___], AAA[___]] :> 0;
  in = in //. NCM[AAA[___], DDD[___], AAA[___]] :> 0;
  in = in //. NCM[AAA[___], AAA[___], DDD[___]] :> 0;
  in = in //. AAA[x_, a_, μ_] ** AAA[z_, b_, ν_] :> Module[ {k = Unique["k"]},
       -i delta8[a, b] 1/(2 π)^DIM INTEG[k] EXP[-i, k, x - z] metric[μ, ν]/k.k ];
  in = in //. DDD[μ_, AAA[x_, a_, ν_]] ** AAA[z_, b_, ω_] :>
     Module[ {k = Unique["k"]},
       -mom[k[μ]] delta8[a, b] 1/(2 π)^DIM INTEG[k] EXP[-i, k, x - z] metric[ν, ω]/k.k ];
  in = in //. AAA[z_, b_, ω_] ** DDD[μ_, AAA[x_, a_, ν_]] :>
     Module[ {k = Unique["k"]},
       mom[k[μ]] delta8[a, b] 1/(2 π)^DIM INTEG[k] EXP[-i, k, x - z] metric[ν, ω]/k.k ];
  in = in //. DDD[μ_, AAA[x_, a_, ν_]] ** DDD[ϵ_, AAA[z_, b_, ω_]] :>
     Module[ {k = Unique["k"]}, -i mom[k[μ]] mom[k[ϵ]] delta8[a, b]
       1/(2 π)^DIM INTEG[k] EXP[-i, k, x - z] metric[ν, ω]/k.k ];
  in = in //. NCM[AAA[___]] :> 0;
  in = in //. NCM[DDD[___]] :> 0
 ]
```

Non-Pert G/A 4D only

■ nonPertContA4D[x_]

```
Clear[nonPertContA4D]

nonPertContA4D[y_] := Module[{in, ρ = Unique["ρ"]},
  in = y //. delta3[α_, β_] λλλ[a_, α_, γ_] :> λλλ[a, β, γ];
  in = in //. delta3[α_, β_] λλλ[a_, γ_, α_] :> λλλ[a, γ, β];
  in = in /. m___ λλλ[a_, α_, β_] λλλ[b_, β_, α_]
       GGG[o, a_, ω_, f_] ** AAA[z_, b_, σ_] :> m -1/(DIM (DIM - 1)) G2 mom[z[ρ]]
       (metric[ω, σ] metric[f, ρ] - metric[ω, ρ] metric[f, σ]);
  in = Expand[in]
 ]
```

■ nonPertExpSetup[x_]

```
Clear[nonPertExpSetup]

nonPertExpSetup[y_] := Module[{in},
  in = y //. EXP[-i, k_, x_ - z_] :> EXP[i, k, z - x];
  in = in //. EXP[i, k_, x_ - z_] :> EXP[i, k, x] EXP[i, -k, z];
  in = in //. EXP[i, k_, o] :> 1;
  in = in //. EXP[-i, k_, o] :> 1;
  in = in //. EXP[i, k_, 0] :> 1;
  in = in //. EXP[-i, k_, 0] :> 1;
  in = in //. EXP[i, a_, x_] EXP[i, b__, x_] :> EXP[i, a + b, x];
  in = in //. INTEG[x] EXP[i, b__, x] :> (2 π)^DIM DDelta4[b];
  in
 ]
```

■ nonPertExpansionMomDeal4D[x_]

```
Clear[nonPertExpansionMomDeal4D]

nonPertExpansionMomDeal4D[y_] := Module[{in, a = Unique["a"], b = Unique["b"]},
  in =
   y //. INTEG[z] INTEG[p_] mom[z[μ_]] EXP[i, p_ - q_, z] SSS[SLASH[p_], r_, i_] :>
     -i (2 π)^DIM SSS[SLASH[q], r, a] γγγ[μ, a, b] SSS[SLASH[q], b, i];
  in = Expand[in]
 ]
```

Non-Pert G/A 4D only (SECOND WAY)

■ nonPertExpansionMomDeal4DOther[x_]

```
Clear[nonPertExpansionMomDeal4DOther]

nonPertExpansionMomDeal4DOther[y_] := Module[{in, ϵ = Unique["ϵ"]},
  in =
   y //. INTEG[z] INTEG[p_] mom[z[ρ_]] EXP[i, p_ - q_, z] SSS[SLASH[p_], r_, i_] :>
     -(i 2^(1+DIM) mass π^DIM mom[q[ρ]] III[r, i])/(-mass^2 + q.q)^2 + (i (2 π)^DIM metric[ϵ, ρ] γγγ[ϵ, r, i])/(-mass^2 + q.q) -
      (i 2^(1+DIM) π^DIM mom[q[ϵ]] mom[q[ρ]] γγγ[ϵ, r, i])/(-mass^2 + q.q)^2;
  in = Expand[in]
 ]
```

Non-Pert G/A 6D only

■ nonPertContA6D[x_]

```
Clear[nonPertContA6D]

nonPertContA6D[y_] :=
 Module[{in, ρ = Unique["ρ"], ρ1 = Unique["ρ"], ρ2 = Unique["ρ"]},
  in = y //. delta3[α_, β_] λλλ[a_, α_, γ_] :> λλλ[a, β, γ];
  in = in //. delta3[α_, β_] λλλ[a_, γ_, α_] :> λλλ[a, γ, β];
  in =
   in /. m___ λλλ[a_, α_, β_] λλλ[b_, β_, α_] GGG[o, a_, ω_, f_] ** AAA[z_, b_, σ_] :>
     m (-1)/2 mom[z[ρ]] mom[z[ρ1]] mom[z[ρ2]] G3 (-1)/(2 DIM (DIM^2 - 4)) gs
      ((2 metric[ρ1, ρ2] (metric[ω, ρ] metric[f, σ] - metric[ρ, f] metric[ω, σ])) +
       (-3)/(DIM - 1) (metric[ρ, ρ2] (metric[ω, ρ1] metric[f, σ] -
            metric[ρ1, f] metric[ω, σ]) - metric[σ, ρ2]
          (metric[ρ, f] metric[ω, ρ1] - metric[ω, ρ] metric[f, ρ1])) +
       (-1) (metric[ρ1, σ] (metric[ω, ρ2] metric[ρ, f] -
            metric[ω, ρ] metric[f, ρ2]) - metric[ρ1, ρ]
          (metric[ω, ρ2] metric[f, σ] - metric[ω, σ] metric[ρ2, f])));
  in = Expand[in]
 ]
```

■ nonPertExpansionMomDeal6D[x_]

```
Clear[nonPertExpansionMomDeal6D]

nonPertExpansionMomDeal6D[y_] := Module[{in, ϵ = Unique["ϵ"]},
  in = y //. INTEG[z] INTEG[p_] mom[z[μ_]]
       mom[z[ν_]] mom[z[ρ_]] EXP[i, p_ - q_, z] SSS[SLASH[p_], r_, i_] :>
     (i 2^(3+DIM) mass π^DIM metric[ν, ρ] mom[q[μ]] III[r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) mass π^DIM metric[μ, ρ] mom[q[ν]] III[r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) mass π^DIM metric[μ, ν] mom[q[ρ]] III[r, i]) / (-mass^2 + q.q)^3 -
      (3 i 2^(4+DIM) mass π^DIM mom[q[μ]] mom[q[ν]] mom[q[ρ]] III[r, i]) / (-mass^2 + q.q)^4 -
      (i 2^(1+DIM) π^DIM metric[ϵ, ρ] metric[μ, ν] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^2 -
      (i 2^(1+DIM) π^DIM metric[ϵ, ν] metric[μ, ρ] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^2 -
      (i 2^(1+DIM) π^DIM metric[ϵ, μ] metric[ν, ρ] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^2 +
      (i 2^(3+DIM) π^DIM metric[ν, ρ] mom[q[ϵ]] mom[q[μ]] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) π^DIM metric[μ, ρ] mom[q[ϵ]] mom[q[ν]] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) π^DIM metric[ϵ, ρ] mom[q[μ]] mom[q[ν]] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) π^DIM metric[μ, ν] mom[q[ϵ]] mom[q[ρ]] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) π^DIM metric[ϵ, ν] mom[q[μ]] mom[q[ρ]] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^3 +
      (i 2^(3+DIM) π^DIM metric[ϵ, μ] mom[q[ν]] mom[q[ρ]] γγγ[ϵ, r, i]) / (-mass^2 + q.q)^3 -
      (3 i 2^(4+DIM) π^DIM mom[q[ϵ]] mom[q[μ]] mom[q[ν]] mom[q[ρ]] γγγ[ϵ, r, i]) /
       (-mass^2 + q.q)^4;
  in = Expand[in]
 ]
```

Integration

δ Integrals

■ doDeltaIntegs[x_]

```
Clear[doDeltaIntegs, deltaIntegs, intSelector, fpFlipInts, flipInts, INTME]

doDeltaIntegs[x_] :=
 If[TrueQ[Head[x] == Plus], Map[deltaIntegs, x], deltaIntegs[x]]

deltaIntegs[x_] := Module[{inInt, inDelt, inRest, numbIntsNeeded, selections},
  inInt = List @@ DeleteCases[x, Except[INTEG[___]]] //. INTEG[a_] :> a;
  inDelt = Apply[List,
    (List @@ DeleteCases[x, Except[DDelta4[___]]] //. DDelta4[a_] :> a), 1];
  inRest = DeleteCases[DeleteCases[x, INTEG[___]], DDelta4[___]];
  numbIntsNeeded = Length[inDelt];
  selections = intSelector[numbIntsNeeded, inInt, inDelt];
  fpFlipInts[x selections]
 ]

intSelector[n_, i_, d_] :=
 Module[{integrateOver = {}, notOptions, maybeSelectFrom, j, alsoNotOpts, k},
  notOptions =
   Append[Cases[Tally[Flatten[d] //. -1 a_ :> a], {x_, m_ /; m > 1} :> x], -1];
  maybeSelectFrom = d;
  For[j = 1, j ≤ Length[notOptions], j++,
   maybeSelectFrom = DeleteCases[maybeSelectFrom, notOptions[[j]], Infinity]];
  alsoNotOpts = Append[DeleteCases[Flatten[maybeSelectFrom],
     Alternatives @@ i], -1];
  For[k = 1, k ≤ Length[alsoNotOpts], k++, maybeSelectFrom =
    DeleteCases[maybeSelectFrom, alsoNotOpts[[k]], Infinity]];
  For[l = 1, l ≤ n, l++, integrateOver =
    Append[integrateOver, maybeSelectFrom[[l]][[1]]]];
  Map[INTME, integrateOver] /. List → Times
 ]

fpFlipInts[y_] := Module[{in},
  in = y;
  While[MemberQ[in, INTME[___]], in = flipInts[in]];
  in
 ]

flipInts[y_] := Module[{flip, out},
  flip = y /. a__ INTME[b_] :> b;
  out = y //.
     INTME[flip] INTEG[flip] DDelta4[-flip + c_ + d_] i_. :> flipIt[flip, c + d] i //.
    INTME[flip] INTEG[flip] DDelta4[flip + c_ + d_] i_. :> flipIt[flip, -c - d] i;
  out = out //. flipIt[x_, a__] i_. :> (i /. x → a)
 ]
```

Fix Denominator for TARCER

■ denomNegFix[x_]

```
Clear[negHandling, flipADenomTerm, allGood, denomNegFix]

allGood[y_, i_, j_] :=
 Module[{memb12, memb1q, memb2q, membNeg12, membNeg1q, membNeg2q, out},
  memb12 = MemberQ[y, i + j];
  membNeg12 = MemberQ[y, -i - j];
  memb1q = MemberQ[y, i + q];
  memb2q = MemberQ[y, j + q];
  membNeg1q = MemberQ[y, -i - q];
  membNeg2q = MemberQ[y, -j - q];
  out = If[memb12 == False && membNeg12 == False && memb1q == False &&
      memb2q == False && membNeg1q == False && membNeg1q == False, True, False]
 ]

flipADenomTerm[y_, i_] := Module[{out},
  out = Replace[y, i :> -i, {1, Infinity}]
 ]

negHandling[y_] :=
 Module[{integList, i, j, denomDots, denomTerms, aGo, aG1, aG2, aGb, out},
  integList = y /. x___ INTEG[a_] INTEG[b_] :> {a, b};
  i = integList[[1]];
  j = integList[[2]];
  denomDots = DeleteCases[
    DeleteCases[DeleteCases[Cases[List @@ (Denominator[y] //. mass → 0), _Dot],
      Dot[integList[[1]], integList[[1]]]],
     Dot[integList[[2]], integList[[2]]]], Dot[q, q]];
  denomTerms = denomDots /. Dot[a_, a_] :> a;
  aGo = allGood[denomTerms, i, j];
  aG1 = allGood[flipADenomTerm[denomTerms, i], i, j];
  aG2 = allGood[flipADenomTerm[denomTerms, j], i, j];
  aGb = allGood[flipADenomTerm[flipADenomTerm[denomTerms, i], j], i, j];
  out = Which[aGo, {}, aG1, {i}, aG2, {j}, aGb, {i, j}];
  out
 ]

denomNegFix[y_] := Module[{flip, flipped},
  If[y == 0, Return[0]];
  flip = negHandling[y[[1]]];
  flipped = Which[TrueQ[Length[flip] == 0], y, TrueQ[Length[flip] == 1],
    Replace[y, flip[[1]] → -flip[[1]], {1, Infinity}], TrueQ[Length[flip] == 2],
    Replace[Replace[y, flip[[1]] → -flip[[1]], {1, Infinity}],
     flip[[2]] → -flip[[2]], {1, Infinity}]];
  flipped = flipped //. INTEG[-a_] :> INTEG[a];
  flipped = flipped //. Dot[-a_, -b_] :> Dot[a, b] //.
     Dot[-a_, b_] :> -Dot[a, b] //. Dot[a_, -b_] :> -Dot[a, b];
  flipped
 ]
```

Integrate

dimReg

Tarcer

■ integTarcer[x_]

```
Clear[integTarcer, ToTARCER, tarcIt, toTFI, negRT]

integTarcer[y_] := Module[{in},
  in = y //. x_ INTEG[p_] INTEG[q_] :> (2 π)^(2 DIM) tarcIt[x, q, p];
  in = in //. x_ INTEG[p_] :> dimReg[x, p];
  in
 ]

ToTARCER[a_ + b_, k1_, k2_] := ToTARCER[a, k1, k2] + ToTARCER[b, k1, k2]

ToTARCER[a_ b_, k1_, k2_] := a ToTARCER[b, k1, k2] /; FreeQ[a, k1] && FreeQ[a, k2]

tarcIt[x_, k1_, k2_] := Module[{inx},
  inx = 1/(4 π)^DIM ToTARCER[x, k1, k2];
  inx = inx //. ToTARCER[y_, k_, q_] :> toTFI[y, k, q]
 ]

toTFI[x_, k_, q_] := Module[
  {p, inArg, inArgN, inNeg, pPot, nPot, dPot, y, numOut, denOut, someOut},
  p =
   DeleteCases[DeleteCases[DeleteDuplicates[Cases[Cases[Denominator[x], Dot[_,
           _], {0, Infinity}], _Symbol, {0, Infinity}]], k], q] /. {pPot_} → pPot;
  inArg = x /. (msq_ - (r_ .1_))^a_. :> (-1)^a ((r.1) - negRT[-msq]^2)^a;
  inNeg = Cases[inArg, -1] /. {y_} → y /. {} → 1;
  If[TrueQ[inNeg == -1], inArg = -inArg];
  nPot = {{k.k, q.q, p.k, p.q, k.q}, {0.0, 0.0, k.p, q.p, q.k}};
  someOut = ∋.∋ inArg;
  numOut = {0, 0, 0, 0, 0};
  Do[numOut[[i]] =
    Cases[Numerator[someOut], (nPot[[1]][[i]])^-] /. (Dot[a_, b_])^c_. :> c /.
      {y_} → y /. {} → 0 + Cases[Numerator[someOut], (nPot[[2]][[i]])^-] /.
     (Dot[a_, b_])^c_. :> c /. {y_} → y /. {} → 0, {i, 5}];
  dPot = {{k.k - msq1_, q.q - msq2_, (k - p).(k - p) - msq3_, (q - p).(q - p) - msq4_,
     (k - q).(k - q) - msq5_}, {0.0 - msq1_, 0.0 - msq2_,
     (p - k).(p - k) - msq3_, (p - q).(p - q) - msq4_, (q - k).(q - k) - msq5_},
    {k.k, q.q, (k - p).(k - p), (q - p).(q - p), (k - q).(k - q)},
    {0.0, 0.0, (p - k).(p - k), (p - q).(p - q), (q - k).(q - k)}};
  denOut = {{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}};
  Do[denOut[[i]][[1]] =
    Cases[Denominator[inArg], (dPot[[1]][[i]])^-] /. (Dot[a_, b_] - msq1_)^c_. :>
                 c /. {y_} → y /. {} → 0 + Cases[Denominator[inArg],
```

```
                (dPot[[2]][[i]])^-.] /. (Dot[a_, b_] - msq1_)^c_-. :> c /.
            {y_} -> y /. {} -> 0 + Cases[Denominator[inArg], (dPot[[3]][[i]])^-.] /.
          (Dot[a_, b_])^c_-. :> c /. {y_} -> y /.
        {} -> 0 + Cases[Denominator[inArg], (dPot[[4]][[i]])^-.] /.
      (Dot[a_, b_])^c_-. :> c /. {y_} -> y /. {} -> 0, {i, 5}];
 Do[denOut[[i]][[2]] = Cases[Denominator[inArg], (dPot[[1]][[i]])^-.] /.
      (Dot[a_, b_] - msq1_)^c_-. :> msq1 /. {(y_)^2} -> y /. {} -> 0 , {i, 5}];
 Do[If[TrueQ[denOut[[i]][[2]] == 0], denOut[[i]][[2]] =
    Cases[Denominator[inArg], (dPot[[2]][[i]])^-.] /.
       (Dot[a_, b_] - msq1_)^c_-. :> msq1 /. {(y_)^2} -> y /. {} -> 0], {i, 5}];
 Times[inNeg, TFI[DIM, p.p, numOut, denOut]]
]
```

■ ABJConv[x_]

```
Clear[aConvRules, bConvRules, jConvRules, ABJConv]

aConvRules = {
    (*TAI[d_,p_,{{a_,0}}]:→This just auto goes to zero probably by tarcer*)
    TAI[d_, p_, {{a_, M_}}] :→ (i)^(1-d) * (-M^2)^(d/2-a) * Γ[a - d/2] / Γ[a]
  };

bConvRules = {
    (*Davydychev equation 4.8 in J Math Phys 32(1991)1052-1060 *)
    TBI[d_, p_, {{a_, M_}, {b_, M_}}] :→ i^(1-d) * (-M^2)^(d/2-a-b) * Γ[a+b-d/2] / Γ[a+b] *
       HPFQ[{a, b, a+b-d/2}, {(a+b)/2, (a+b+1)/2}, p/(4 M^2)] /; UnsameQ[M, 0]
  };

jConvRules = {
    (*Broadhurst 9304303v1 27 Apr 1993 but we Wick rotated and after Jacobian
     transform this to take it from Euclidean to Minkowski space.*)
    TJI[d_, p_, {{a_, M_}, {b_, M_}, {c_, 0}}] :→
     (-1)^(1-d) * (-M^2)^(d-a-b-c) * Γ[a+b+c-d] Γ[d/2 - c] Γ[b+c-d/2] Γ[a+c-d/2] /
       (Γ[a] Γ[b] Γ[d/2] Γ[a+b+2c-d]) *
       HypergeometricPFQ[{b, a+b+c-d, b+c-(d/2), a+c-(d/2)},
        {d/2, c+(a+b-d)/2, c+(a+b-d+1)/2}, p/(4 M^2)] /; UnsameQ[M, 0],
    (*Massless like Broadhurst just made by 2 dimRegs*)
    TJI[d_, p_, {{a_, 0}, {c_, 0}, {b_, 0}}] :→
     -((-1)^(2ϵ) (p)^(4-a-b-c+2ϵ) Gamma[-4+a+b+c-2ϵ] Gamma[2-a+ϵ] Gamma[2-b+ϵ]
         Gamma[2-c+ϵ]) / (Gamma[a] Gamma[b] Gamma[c] Gamma[6-a-b-c+3ϵ])
  };

ABJConv[x_] := Simplify[x /. aConvRules /. bConvRules /. jConvRules /. DIM → dim];
```

Trace Handling

■ doTraces[x_]...(7)

```
Clear[gamma5deal, doTraces, gamma5trace, normalGammaTrace,
 jTrace, takeTrace, getThe5, getThe7, getThe9, g5p8, takeG5trace]
gamma5deal[x_] :=
 Module[{a, b, c, d, e, f, f2, f5, g, h, i, j},
  a = Cases[x, dm[γ[μ_]] → μ, Infinity];
  b = SortBy[a, Real];
  c = DeleteCases[a, 5];
  d = Total[Flatten[Position[a, 5] - Position[b, 5]]];
  e = If[ OddQ[Count[a, 5]], Append[DeleteCases[a, 5], 5], DeleteCases[a, 5]];
  f = (-1)^d;
  f2 = (-1)^Length[DeleteCases[a, 5]];
  f5 = If[MemberQ[e, 5], f * f2, f];
  g = NCM @@ Cases[e, μ_ → dm[γ[μ]], Infinity] * f5;
  h = If[MemberQ[x, dm[γ[5]]], g, x];
  h
 ]
jTrace[NCM[dm[γ[μ_]], dm[γ[ν_]]]] := 4 metric[μ, ν]
jTrace[x_] :=
 Sum[(-1)^i metric[x[[1]] /. dm[γ[ρ_]] :> ρ, x[[i]] /. dm[γ[ρ_]] :> ρ]
   jTrace[DeleteCases[x, x[[1]] | x[[i]]]], {i, 2, Length[x]}]
takeTrace[x___] := (If[EvenQ[Length[x]] == False, Return[0]]; jTrace[x])
takeG5trace[x___] := Which[
  Length[x] == 3, 0,
  EvenQ[Length[x]], 0,
  Length[x] == 5, getThe5[x],
  Length[x] == 7, getThe7[x],
  Length[x] == 9, getThe9[x],
  OddQ[Length[x]], PROBLEMin[takeγ5Tr]
 ]
gamma5trace[x_] := Module[{a},
  a = Expand[takeG5trace[x]];
  a
 ]
normalGammaTrace[x_] := Module[{in},
  in = Expand[takeTrace[x]];
  in
 ]
doTraces[x_] :=
 Module[{in, y},
  in = Expand[Distribute[x]] //. NCM[y_] :> gamma5deal[NCM[y]];
  in = in //. NCM[y_] :> gamma5trace[NCM[y]] /; MemberQ[y, dm[γ[5]]];
  in = in //. NCM[y_] :> normalGammaTrace[NCM[y]] /; FreeQ[y, dm[γ[5]]];
  in = Expand[in];
  in
 ]
```

Pre-calculated $\gamma 5$ trace results

Projections

■ Π0[x_] or Π1[x_]

```
Clear[Π0, Π1, projectionCleaner]

projectionCleaner[x_] := Module[{in},
  in = x //. metric[μ_, ν_] mom[p_[μ_]] :→ mom[p[ν]];
  in = in //. mom[p_[μ_]] mom[q_[μ_]] :→ Dot[p, q];
  in = in //. mom[p_[μ_]]^2 :→ Dot[p, p];
  in = in //. metric[μ_, ν_]^2 :→ DIM;
  in
 ]

(*Scalar*)
Π0[x_, q_: q, μ_: μ, ν_: ν] := projectionCleaner[Expand[mom[q[μ]] mom[q[ν]] / (q.q) x]];

(*Vector*)
Π1[x_, q_: q, μ_: μ, ν_: ν] :=
  projectionCleaner[Expand[(mom[q[μ]] mom[q[ν]] - q.q metric[μ, ν]) / ((DIM - 1) (q.q)) x]];
```

Expansion in ϵ

expandCorInEpsilon[x_,y_]...(Last)

```
Clear[expandCorInEpsilon]

expandCorInEpsilon[yExp_, zExp_] := Module[
  {In, ExpIn, HypGeomsIn, HypeCoeffsIn, NonHypeCoeffIn, ExpandedNonHypeCoeffIn,
   ExpandedHypeCoeffsIn, OrderOfHypExpNeededIn, ExpandedHypGeomsIn,
   TotalIn, SortedTotalIn, NonZeroIn, AdjustedNonZeroIn, FinalIn},
  In = Expand[zExp * yExp];
  ExpIn = Expand[In //. q.q/(4 mass^2) → z];
  HypGeomsIn =
   DeleteDuplicates[Cases[ExpIn, HypergeometricPFQ[_, _, _], Infinity]];
  HypeCoeffsIn = Map[Simplify[Coefficient[ExpIn, #]] &, HypGeomsIn];
  NonHypeCoeffIn = Expand[Simplify[ExpIn - Total[HypeCoeffsIn * HypGeomsIn]]];
  ExpandedNonHypeCoeffIn =
   Map[Expand[Normal@Series[#, {ϵ, 0, 0}]] &, NonHypeCoeffIn];
  ExpandedHypeCoeffsIn = Map[Expand[Normal@Series[#, {ϵ, 0, 0}]] &,
    HypeCoeffsIn];
  OrderOfHypExpNeededIn = Map[
    (-1) Min[Cases[DeleteDuplicates[Cases[#, Power[ϵ, _], Infinity]],
        _Integer, Infinity]] &, ExpandedHypeCoeffsIn];
  ExpandedHypGeomsIn = Map[Expand, MapThread[HypExp,
     {HypGeomsIn, Table[ϵ, {Count[HypGeomsIn, _HypergeometricPFQ, Infinity]}],
      OrderOfHypExpNeededIn}]];
  TotalIn = Total[Expand[ExpandedHypGeomsIn * ExpandedHypeCoeffsIn]] +
    ExpandedNonHypeCoeffIn;
  SortedTotalIn = Collect[TotalIn, ϵ, Simplify];
  NonZeroIn = SortedTotalIn //. {Coefficient[SortedTotalIn, ϵ, 1] → 0,
     Coefficient[SortedTotalIn, ϵ, 2] → 0};
  AdjustedNonZeroIn = NonZeroIn //. {gs^2 → 4 π αs, Log[mass^2] → Log[mass^2/ν^2] +
       2 Log[ν], Log[-mass^2] → Log[-mass^2/ν^2] + 2 Log[ν], q.q → 4 z mass^2};
  FinalIn = Collect[AdjustedNonZeroIn, ϵ, Simplify];
  FinalIn
 ]
```